Modeling by the nonlinear stochastic differential equation of the power-law distribution of extreme events in the financial systems

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with
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Focus of the talk

Our researches are related with the

Nonlinear stochastic differential equations (SDE),

- **resulting in power-law distributions,**
- **including the Inverse Cubic Law**
- **1/f noise and**
- **bursting processes**

Inverse Cubic Law

One of *stylized facts* emerging from statistical analysis of *financial markets* is the *inverse cubic law* for the *cumulative* distribution of a number of events of trades and of the logarithmic price change.

- P. Gopikrishnan, M. Meyer, L. A. N. Amaral, H. E. Stanley, *Eur. Phys. J. B*, 3, p. 139, 1998.
- S. Solomon and P. Richmond, *Physica A*, 299, p. 188, 2001.
- X. Gabaix, P. Gopikrishnan, V. Plerou, H. E. Stanley, *Nature*, 423, p.267, 2003.
- B. Podobnik, D. Horvatic, A. M. Petersen, H. E. Stanley, *PNAS*, 106, p. 22079, 2009.
- G.-H. Mu and W.-X. Zhou, *Phys. Rev. E*, 82, 066103, 2010.

Examples of the Inverse Cubic Law

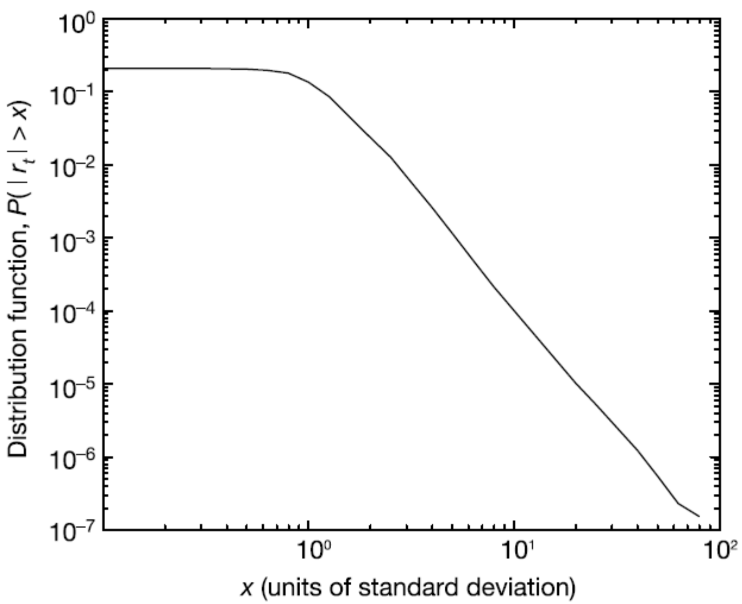


Figure 1 Cumulative distributions of the normalized 15-min absolute returns of the 1,000 largest companies in the 'Trades and Quotes' database for the 2-yr period 1994–1995. We define the normalized return as $r_{it} = (\tilde{r}_{it} - \tilde{r}_i)/\sigma_i$, where $\tilde{r}_i$ and σ_i are the mean and the standard deviation of the unnormalized return $\tilde{r}_{it}$ of stock *i*. We obtain $P(|r_t| > x) \sim x^{-\zeta_r}$ with $\zeta_r = 3.1 \pm 0.1$.

X. Gabaix, P. Gopikrishnan, V. Plerou, H. E. Stanley, *Nature*, 423, p.267, 2003.

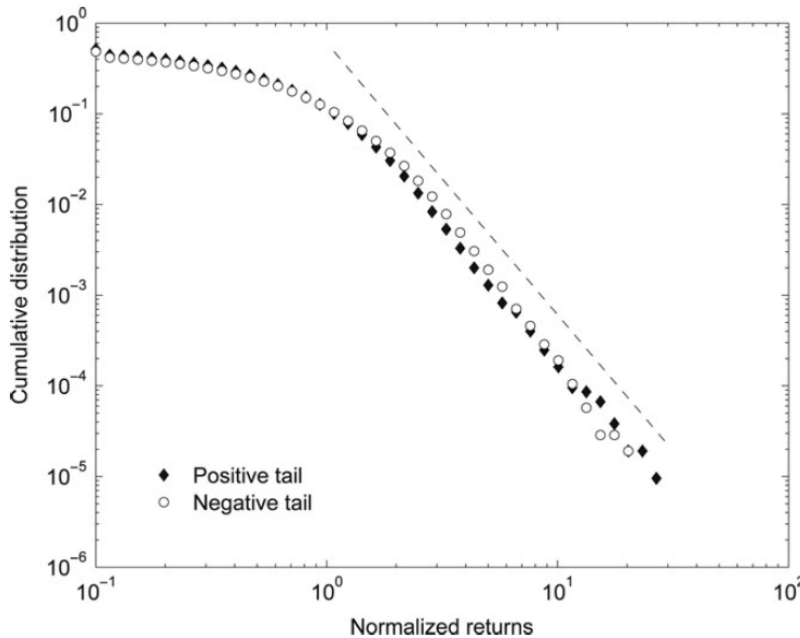
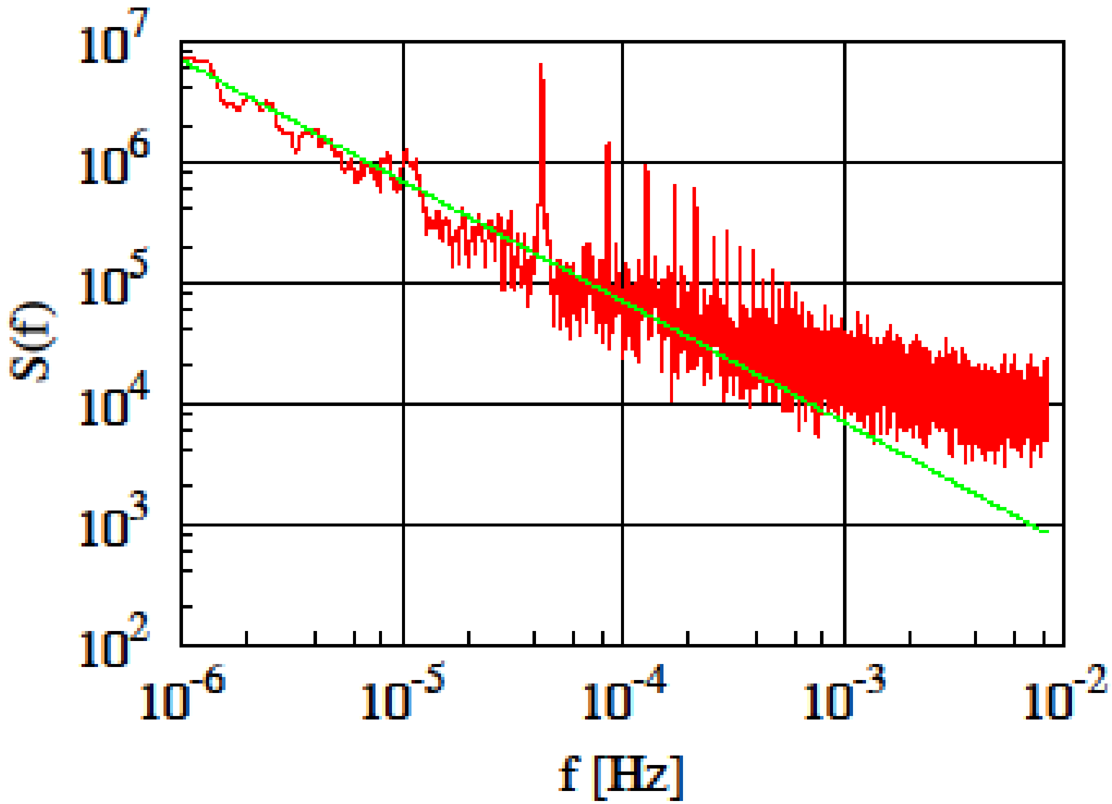


Fig. 3. The cumulative distribution of the normalized 1-min return for the NSE Nifty index. The broken line indicates a power law with exponent $\alpha = 3$.

R.K.Pan, S.Sinha, *Physica A*, 387, p.495, 2008.

B. Kaulakys, Vilnius University, Lithuania: **www.itpa.lt/kaulakys**

It is the long-range process with 1/f fluctuations

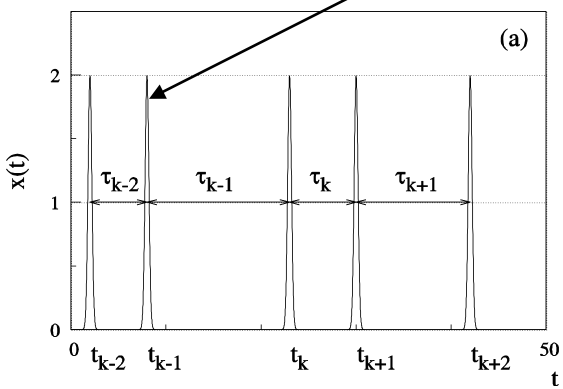


Power spectral density of trading activity

(number of trades per 1 min.) for ABT stock on NYSE

Starting from the autoregressive point process model

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma \tau_k^{\mu} \varepsilon_k .$$



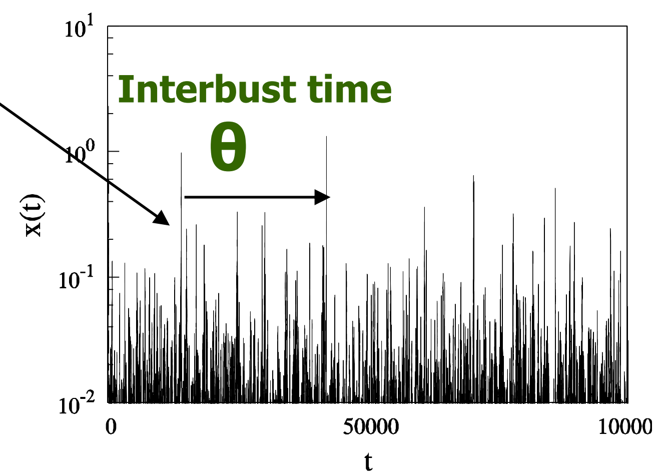
we derive a class of the nonlinear stochastic differential equations

$$\frac{dx}{dt_s} = \Gamma x^{2\eta-1} + x^{\eta} \xi(t_s)$$

which generate bursting, power-law distributed, q-exp and q-Gaussian signals and $1/f^{\beta}$ noise

$$P(x) \sim \frac{1}{x^{\lambda}}, \qquad \lambda = 2(\eta - \Gamma)$$

$$S(f) \sim \frac{1}{f^{\beta}}, \qquad \beta = 2 - \frac{2\Gamma + 1}{2\eta - 2}.$$



The signal

THE POINT PROCESS MODEL

The signal of the model consists of pulses or events

$$I(t) = \sum_k A_k(t - t_k)$$

In a low frequency region and for long-range correlations we can restrict analysis to the noise originated from the correlations between the occurrence times t_k.

Therefore, we can simplify the signal to the point process

The point process

$$I(t) = \bar{a} \sum_k \delta(t - t_k)$$

is primarily and basically defined by the occurrence times $t_1, t_2, \ldots t_k \ldots$

Or by the interevents times $\tau_k = t_{k+1} - t_k$

Stochastic multiplicative point process

Quite generally the dependence of the mean interpulse time on the occurrence number *k* may be described by the general Langevin equation with the drift coefficient $d(\tau_k)$

and a multiplicative noise $b(\tau_k)\xi(k)$

$$\frac{d\tau_k}{dk} = d(\tau_k) + b(\tau_k)\,\xi(k) \,.$$

Replacing the averaging over *k* by the averaging over the distribution of the interpulse times $\tau_k,\ P_k(\tau_k),$ **we have the power spectrum**

$$S(f) = 4\bar{I}^2\bar{\tau}\int_0^\infty d\tau_k P_k(\tau_k)\,\mathrm{Re}\int_0^\infty dq \exp\left\{i\omega\left[\tau_k q + d(\tau_k)\frac{q^2}{2}\right]\right\}$$

$$= 2\bar{I}^2\frac{\bar{\tau}}{\sqrt{\pi}f}\int_0^\infty P_k(\tau_k)\,\mathrm{Re}\left[e^{-i\left(x-\frac{\pi}{4}\right)}\,\mathrm{erfc}\,\sqrt{-ix}\right]\frac{\sqrt{x}}{\tau_k}d\tau_k$$

✓**B. K., et all. Phys. Rev. E 71,** 051105 **(2005)**

Nonlinear stochastic differential equation generating 1/f noise

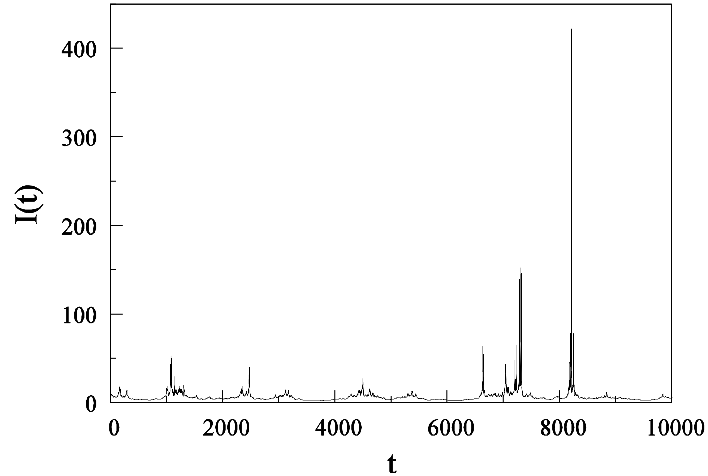
$$\tau_{k+1} = \tau_k + \sigma\varepsilon_k, \; S(f) \propto 1/f$$

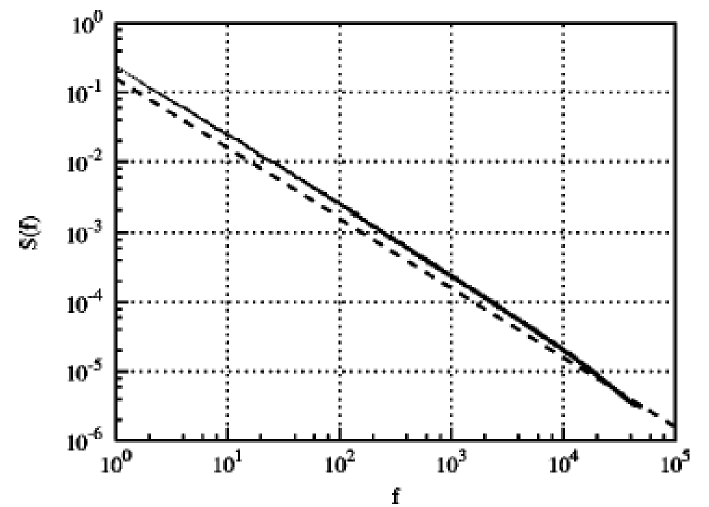
$$\frac{d\tau_k}{dk} = \sigma\xi(k) \quad \langle \xi(k)\xi(k')\rangle = \delta(k-k')$$

$$\mathrm{d}t = \tau_k \mathrm{d}k, \qquad x = a/\tau_k$$

$$\frac{dx}{dt} = x^4 + x^{5/2}\xi(t), \; S(f) \propto 1/f$$

$$P(x) \sim \frac{1}{x^3}$$





1/*f* noise and power-law distribution

✓**B. K. and J. Ruseckas, Phys. Rev. E 70,** 020101(R) **(2004)**

Therefore, the simplest iterative equation

$$\tau_{k+1} = \tau_k + \sigma \varepsilon_k$$

(with the appropriate boundary conditions)

generating the pure 1/f noise,

corresponds to the *inverse squared*

$$P_{>}(x) \sim x^{-2}$$

***cumulative* distribution.**

We search for the simplest stochastic differential equation,

generating the long-range processes

with the *inverse cubic cumulative distribution.*

The simplest equations generating *the inverse cubic law* of the cumulative distribution, $P_{>}(x) \sim x^{-3}$, are

$$\tau_{k+1} = \tau_k + \sigma \tau_k^{-1/2} \varepsilon_k \,, \tag{8}$$

$$d\tau(t) = \frac{\sigma}{\tau(t)} dW \,, \tag{9}$$

and

$$d\tau(t) = \sigma_x x(t) \, dW \,, \tag{10}$$

where $x(t) = a/\tau(t)$ and $\sigma_x = \sigma/a$.

Equation (10) reveals the particularly obvious meaning, i.e., *the intensity of fluctuations of the interevent time* $\tau(t)$ *is proportional to the intensity of the process* $x(t) \propto 1/\tau(t)$.

The cumulative distribution $P_>(x)$ of x is

$$P_>(x) = \int_x^\infty P(x)\,dx$$
$$\simeq \mathrm{erf}\left(\frac{x_{\min}}{x}\right) - \frac{2x_{\min}}{\sqrt{\pi}x}\exp\left(-\frac{x_{\min}^2}{x^2}\right) \qquad (14)$$
$$= \frac{x_{\min}^3}{x^3}\gamma^*\left(\frac{3}{2}, \frac{x_{\min}^2}{x^2}\right).$$

Here $\gamma^*(a, z)$ is the regularized lower incomplete gamma function. Consequently

$$P_>(x) \simeq \frac{4x_{\min}^3}{3\sqrt{\pi}x^3}, \qquad x \gg x_{\min}, \qquad (15)$$

and we find out *the inverse cubic law.* **Inverse cubic**

Further we can consider a more realistic model assuming that τ_k is a time-dependent average interevent time of the Poissonian-like process with the time-dependent rate. Within this assumption the actual interevent time τ_j is given by the conditional probability [17], [22]

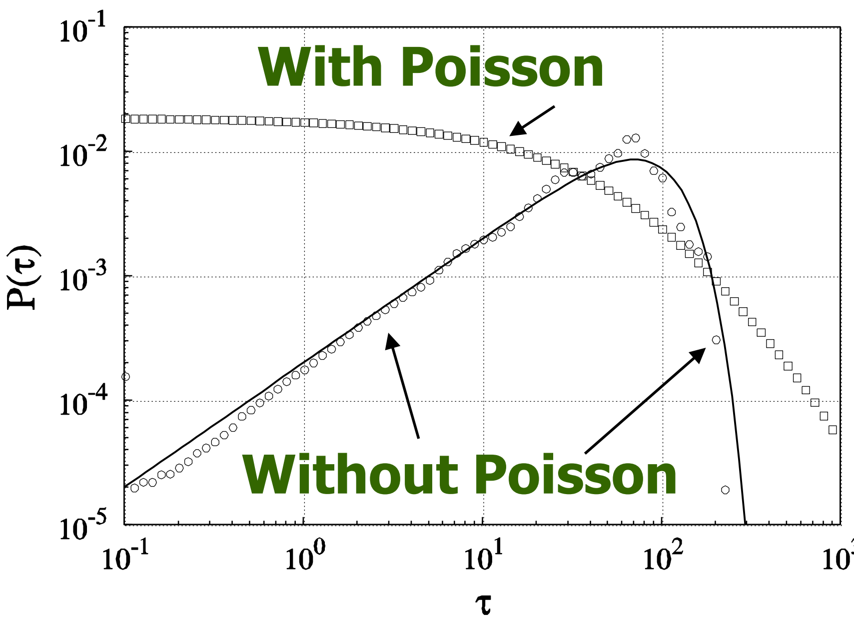
$$\varphi(\tau_j|\tau_k) = \frac{1}{\tau_k} e^{-\tau_j/\tau_k} , \qquad (16)$$

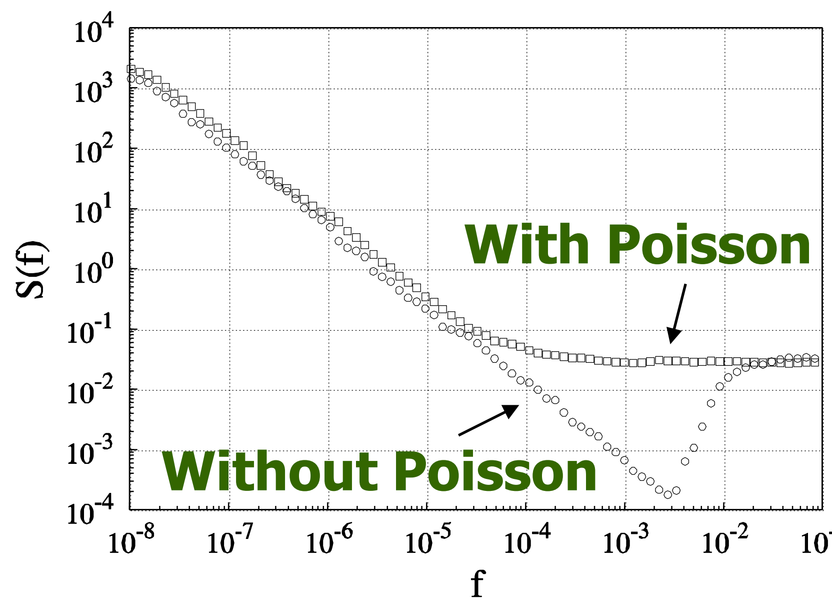
similar to the non-homogeneous Poisson process. In such a case, the distribution of the actual interevent time τ_j is expressed analogically to the superstatistical schemes [30],

$$P_j(\tau_j) = \int \varphi(\tau_j|\tau_k) P_k(\tau_k) d\tau_k \, . \qquad (17)$$

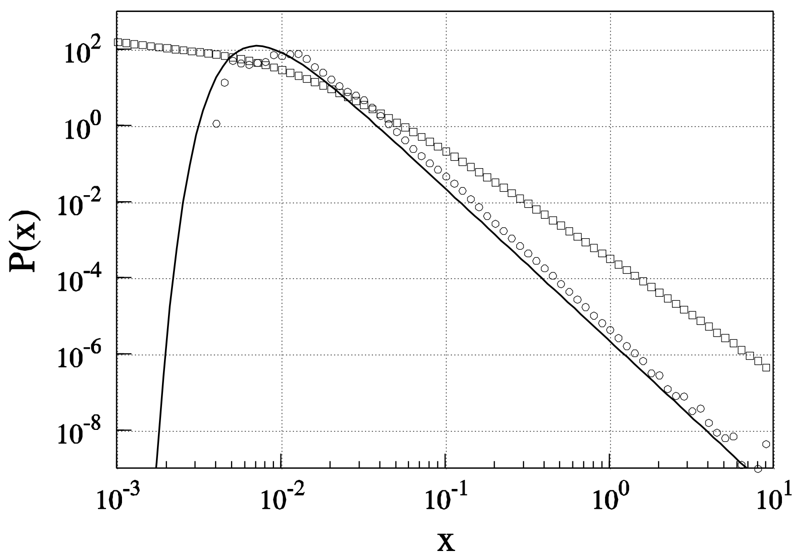
The generalized model (16) and (17) represents a more realistic situation, because the concrete event occurs at random time (like in the Poisson case), however, the average interevent time is slowly (Brownian-like) modulated.

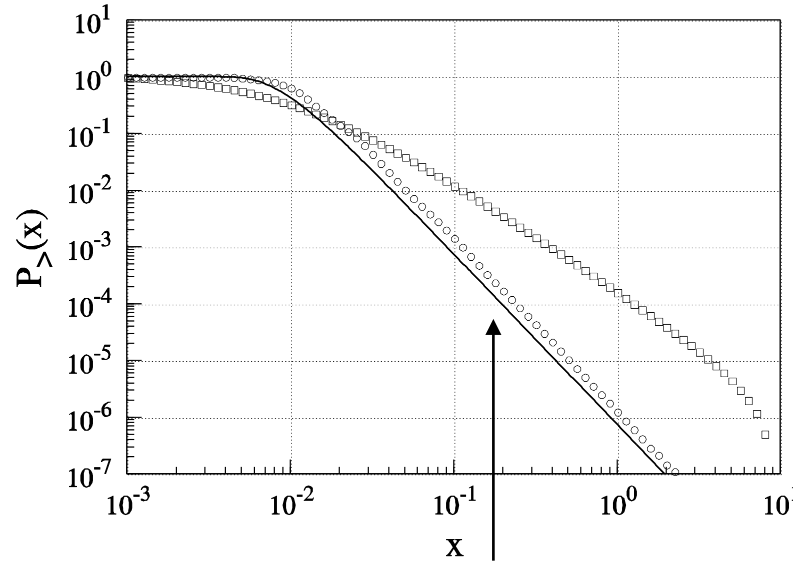
This additional stochasticity of the actual interevent time τ_j by randomization (16) of the concrete occurrence times does not influence on the low frequency power spectra of the signal.





Distribution of the signal depends, however, on the additional Poissonian-like stochasticity





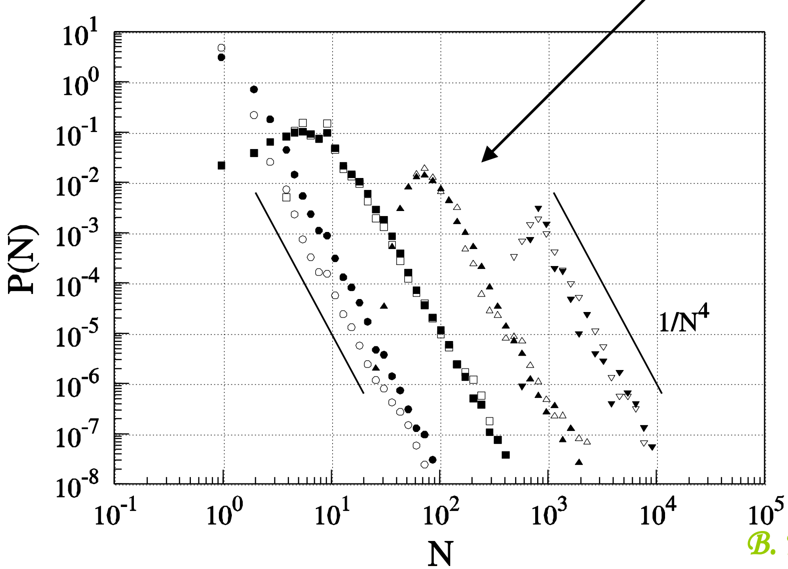
Inverse cubic law

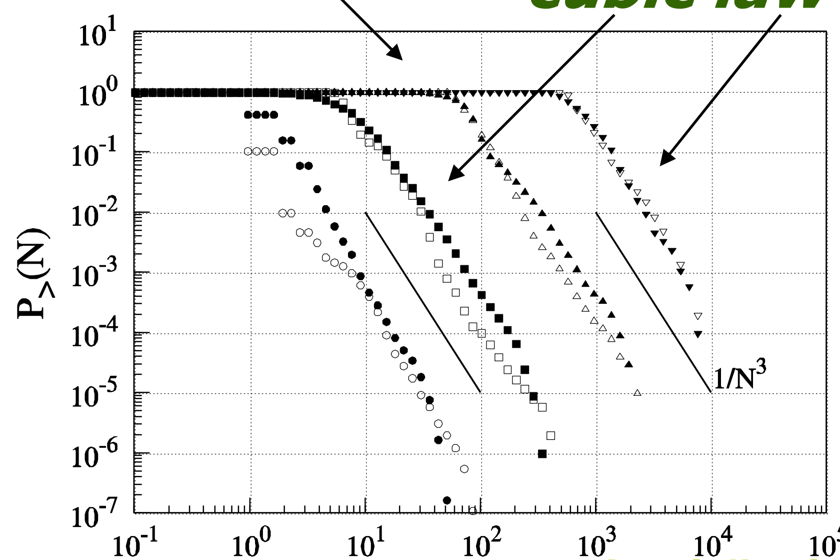
Variable $x(t) = 1/\tau(t)$ represents the formal instantaneous process and does not contain any scale of time. Actually, one measures the number of events N in the definite time window τ_w, e.g., the trading activity, as a number of events in some time interval, or the return at time lag τ_w. These quantities are represented as the integral of the variable $x(t)$ in time interval

$$N(t) = \int_{t}^{t+\tau_w} x(t')\, dt'$$

Does not depend on the additional Poissonian stochasticity

We obtain the *Inverse cubic law*





Generalization of the model

For modeling the long-range processes with $\beta < 1$ and with the power-law correlation function [20]

$$C(t) \sim \frac{1}{t^{1-\beta}} \qquad (19)$$

we should modify Eqs. (8)-(10) assuming the simple additive Brownian motion of small interevent time, keeping the same dependence for large $\tau(t)$. For this purpose, instead of (9) we propose equation

$$d\tau = \sigma \frac{1}{\tau_c + \tau} dW , \qquad (20)$$

where τ_c is a crossover parameter, separating the two kinds of the stochastic motion: (i) the simple Brownian motion for $\tau \ll \tau_c$ and (ii) the model of Section II for $\tau \gg \tau_c$.

Eq. (20) with restrictions at $\tau = \tau_{\min}$ and at $\tau = \tau_{\max}$

$$d\tau = \sigma^2 \left(\frac{\tau_{\min}^2}{\tau^2} - \frac{\tau^2}{\tau_{\max}^2} \right) \frac{dt}{\tau\left(\tau_c + \tau\right)^2} + \sigma \frac{dW}{\tau_c + \tau} . \qquad (21)$$

may be solved using a variable step of integration

$$\Delta t_i = \frac{\kappa^2}{\sigma^2} \left(\tau_c + \tau_i\right)^2 \tau_i^2 \, , \; \kappa \ll 1, \qquad (22)$$

$$\tau_{i+1} = \tau_i + \kappa^2 \left(\frac{\tau_{\min}^2}{\tau_i^2} - \frac{\tau_i^2}{\tau_{\max}^2} \right) \tau_i + \kappa \tau_i \varepsilon_i \, . \qquad (23)$$

The steady-state distribution density $P_k\left(\tau_k\right)$ in k -space of interevent time τ_k, instead of (12), for $\tau_{\min} \ll \tau_c \ll \tau_{\max}$ is

$$P_k\left(\tau_k\right) \simeq \frac{2\left(\tau_c + \tau_k\right)^2}{\tau_{\max}^2 \tau_k} \exp\left(-\frac{\tau_{\min}^2}{\tau_k^2} - \frac{\tau_k^2}{\tau_{\max}^2} \right) . \qquad (24)$$

The steady-state distribution of the intensity of the process $x(t)$, exponentially restricted at small $x_{\mathrm{min}} = 1/\tau_{\mathrm{max}}$ and large $x_{\mathrm{max}} = 1/\tau_{\mathrm{min}}$, is

$$P(x) \simeq \frac{4x_{\mathrm{min}}^3 (x_c + x)^2}{\sqrt{\pi} x^4} \exp\left(-\frac{x_{\mathrm{min}}^2}{x^2} - \frac{x^2}{x_{\mathrm{max}}^2}\right). \quad (25)$$

The cumulative distribution $P_>(x)$ of x for $x < x_c$ is given by the same Eq. (14). The average intensity of the process $\langle x \rangle = \langle \tau_k \rangle^{-1}$, where $\langle \tau_k \rangle \simeq \frac{\sqrt{\pi}}{2} \tau_{\mathrm{max}}$. The counting of events may be calculated according to the same Eq. (18).

✓ **B.K. and M. Alaburda, ICNF'2011 (Toronto)**

The numerical calculations of the power spectral density $S(f)$ of the signal $x(t)$ (1) calculated according to Eqs. (21)–(23) are presented in Fig. 7. The cumulative distributions of this generalization are similar to those of Fig. 4 and Fig. 6.

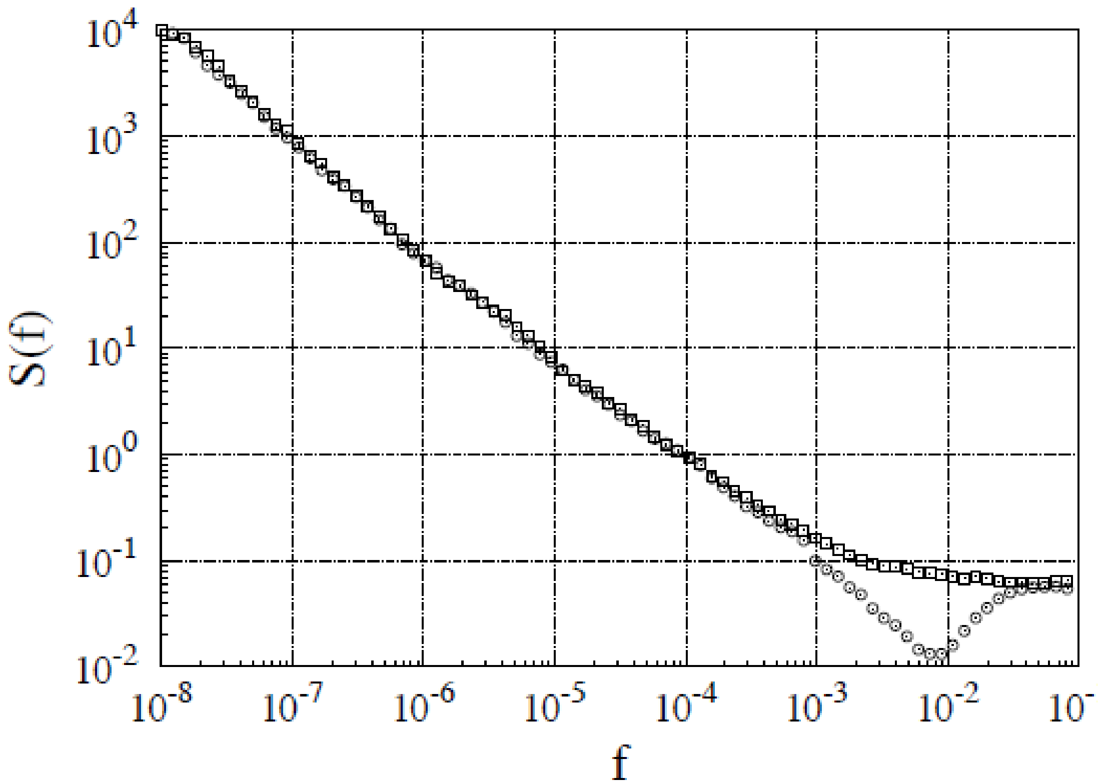


Fig. 7. Power spectral density $S(f)$ of the signal $x(t)$ (1) calculated according to Eqs. (21)–(23), open circles, and that of the Poissonian-like distributed (16) interevent time τ_j, open squares. Used parameters are $\tau_{\min} = 0.01$,

q-exponential distribution

$$dx = \left(\eta - \frac{1}{2}\lambda\right)(x_m + x)^{2\eta-1}\,dt + (x_m + x)^{\eta}\,dW$$

(i) is linear for small x << x_m ,

(ii) restrict divergence of power-law distribution of x at x=0

$$P(x) = \frac{(\lambda-1)x_m^{\lambda-1}}{(x_m+x)^{\lambda}} = \frac{(\lambda-1)}{x_m}\exp_q\left\{-\lambda\frac{x}{x_m}\right\}, \quad x > 0$$

q-exponent

and

(iii) generate signals with 1/f $^{\beta}$ spectrum:

Analytical calculations from the related point process model

$$S(f) \approx \frac{A}{f^{\beta}}, \quad \frac{1}{2} < \beta < 2, \quad 4-\eta < \lambda < 1+2\eta,$$

$$\beta = 1 + \frac{\lambda-3}{2(\eta-1)}, \quad \eta > 1,$$

$$A \approx \frac{(\lambda-1)\,\Gamma(\beta-1/2)\,x_m^{\lambda-1}}{2\sqrt{\pi}\,(\eta-1)\sin(\pi\beta/2)}\left(\frac{2+\lambda-2\eta}{2\pi}\right)^{\beta-1}$$

B. K. and M. Alaburda, J. Stat. Mech. P02051 (2009)

q-Gaussian distribution

$$\mathrm{d}x = \left(\eta - \frac{1}{2}\lambda\right)(x_m^2 + x^2)^{\eta-1} x\mathrm{d}t + (x_m^2 + x^2)^{\eta/2}\mathrm{d}W, \quad \eta > 1, \quad \lambda > 1$$

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\left(\frac{x_m^2}{x_m^2 + x^2}\right)^{\lambda/2} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\exp_q\left\{-\lambda\frac{x^2}{2x_m^2}\right\}$$

q-Gauss

Regular distribution of signal for $x > 0$, $x = 0$ and $x < 0$.

J.Ruseckas and B.K., Phys. Rev. E 84, 051125 (2011).

$$S(f) = \frac{A}{(f_0^2 + f^2)^{\beta/2}} = \exp_q\left\{-\beta\frac{f^2}{2f_0^2}\right\}, \quad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}$$

$$C(s) = \int_0^\infty S(f)\cos(2\pi f s)\mathrm{d}f = \frac{A\sqrt{\pi}}{\Gamma(\beta/2)}\left(\frac{\pi s}{f_0}\right)^h K_h(2\pi f_0 s)$$

$$F(s) = F_2^2(s) = \left\langle |x(t+s) - x(t)|^2 \right\rangle = 2[C(0) - C(s)] = 4\int_0^\infty S(f)\sin^2(\pi s f)\mathrm{d}f.$$

Superstatistical framework

In superstatistical approach the distribution $P(x)$ of the signal x is a superposition of the conditional distributio $\varphi(x|\bar{x})$ and the local stationary distribution $p(\bar{x})$ of the parameter $\bar{x}$,

$$P(x) = \int_0^\infty \varphi(x|\bar{x}) p(\bar{x}) d\bar{x} \, .$$

In order to obtain q-exponential PDF of the signal x we consider exponential PDF, conditioned to the local average value of the parameter $\bar{x}$,

$$\varphi(x|\bar{x}) = \bar{x}^{-1} \exp(-x/\bar{x}) \, .$$

J. Ruseckas and B.K., Phys. Rev. E 84, 051125 **(2011).**

SDE with exponential restriction of diffusion

$$d\bar{x} = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{1}{2}\frac{x_0}{\bar{x}} \right] \bar{x}^{2\eta-1} dt + \sigma \bar{x}^{\eta} dW$$

generates PDF for $\bar{x}$

$$p(\bar{x}) = \frac{1}{x_0 \Gamma(\lambda - 1)} \left(\frac{x_0}{\bar{x}} \right)^{\lambda} \exp\left(-\frac{x_0}{\bar{x}} \right)$$

and *q*-exponential distribution of the signal,

$$P(x) = \frac{\lambda - 1}{x_0} \left(\frac{x_0}{x + x_0} \right)^{\lambda} = \frac{\lambda - 1}{x_0} \exp_q(-\lambda x / x_0) \,, \qquad q = 1 + 1/\lambda \,.$$

By analogy equations $$P(x) = \int_0^\infty \varphi(x|\bar{x}) p(\bar{x}) d\bar{x}$$

$$\varphi(x|\bar{x}) = \frac{1}{\sqrt{\pi}\bar{x}} \exp(-x^2/\bar{x}^2)$$

$$d\bar{x} = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{x_0^2}{\bar{x}^2}\right] \bar{x}^{2\eta-1} dt + \sigma \bar{x}^\eta dW$$

$$p(\bar{x}) = \frac{1}{x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \left(\frac{x_0}{\bar{x}}\right)^\lambda \exp\left(-\frac{x_0^2}{\bar{x}^2}\right)$$

yield *q*-Gaussian distribution

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi} x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \left(\frac{x_0^2}{x_0^2 + x^2}\right)^{\frac{\lambda}{2}} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi} x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \exp_q\left(-\lambda \frac{x^2}{2x_0^2}\right),$$

$$q = 1 + 2/\lambda\,.$$

Nonlinear SDE

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1}\mathrm{d}t_s + x^{\eta}\mathrm{d}W_s.$$

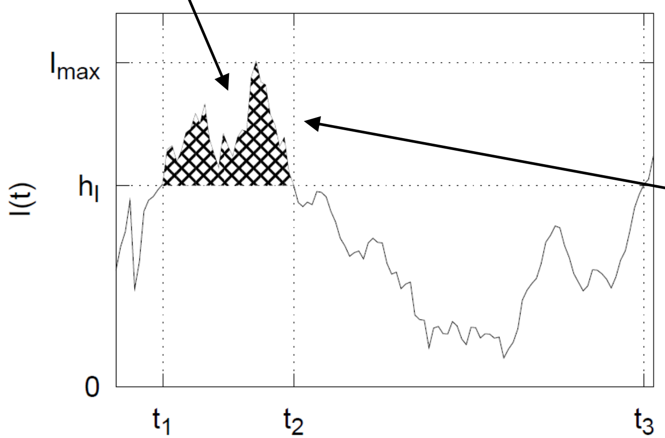
reveal bursting process

Numerical solutions

$$x_{i+1} = x_i + \kappa^2 \left(\eta - \frac{\lambda}{2} + \frac{1}{x_i^2}\right) x_i + \kappa\sqrt{x_i}\zeta_i ,$$

$$t_{s,\,i+1} = t_i + \frac{\kappa^2}{x^{2\eta-2}} ,$$

Area S of burst



bursting process

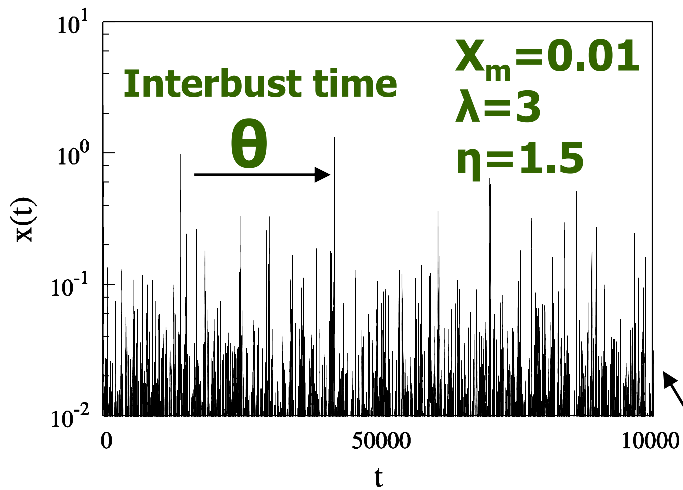
with $1/f^{\beta}$ noise

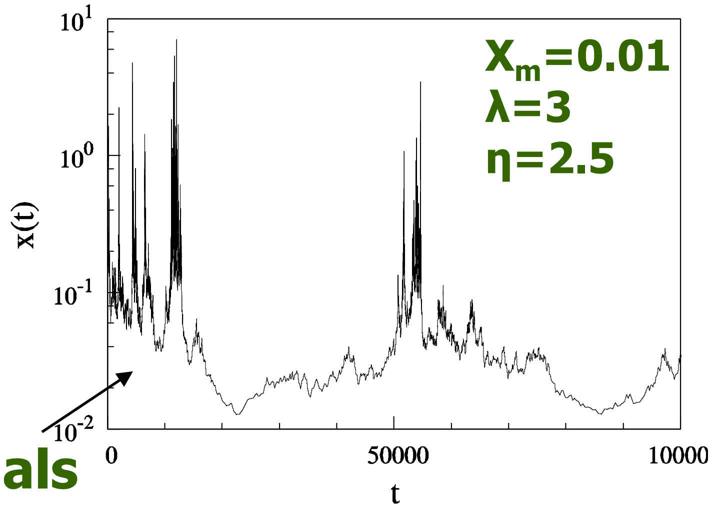
$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}$$

$$T = t_2 - t_1,\ \theta = t_3 - t_2,$$

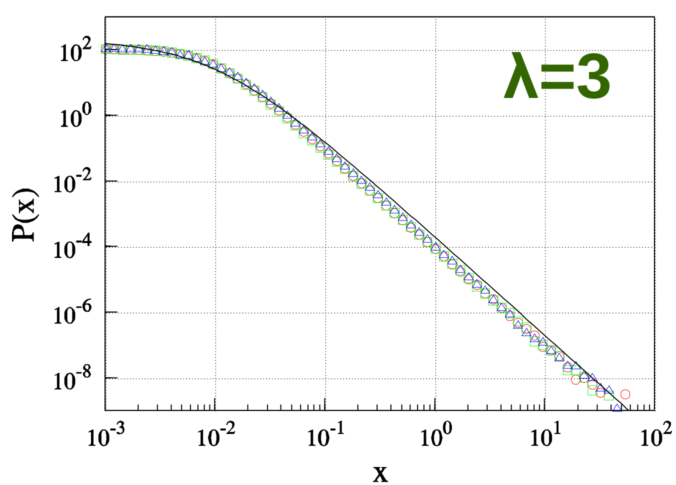
$$\tau = T + \theta = t_3 - t_1$$

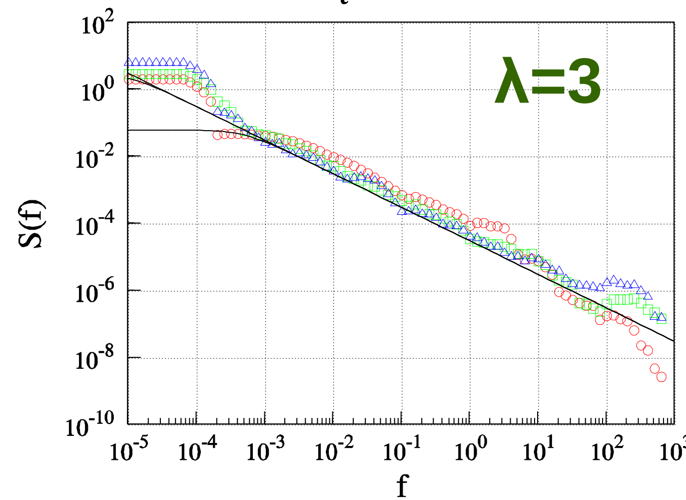
Numerical results. Secondary structure the signals





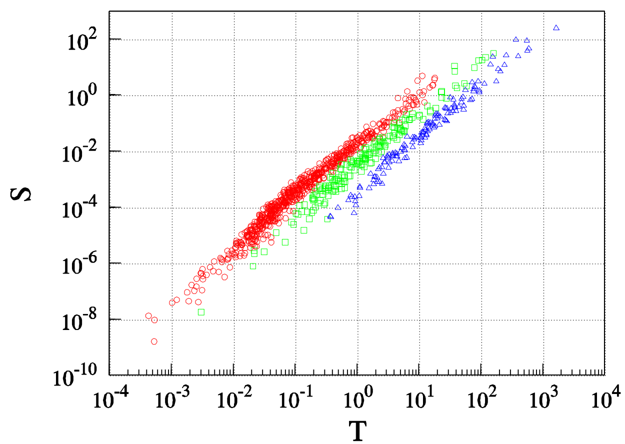
Signals



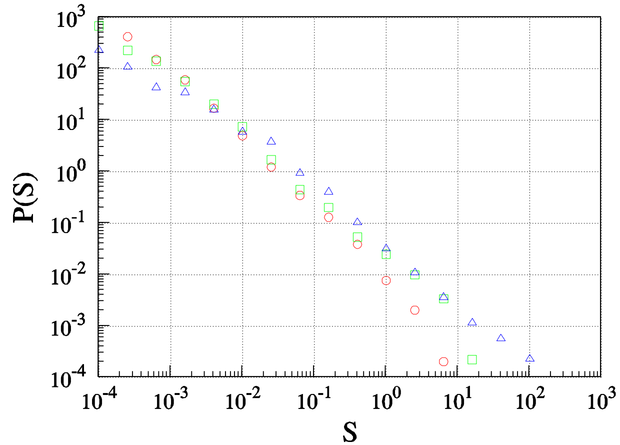


$\eta = 1.5$ (circles), $\eta = 2$ (squares) and $\eta = 2.5$ (triangles) in comparison with the analytical results (solid lines)

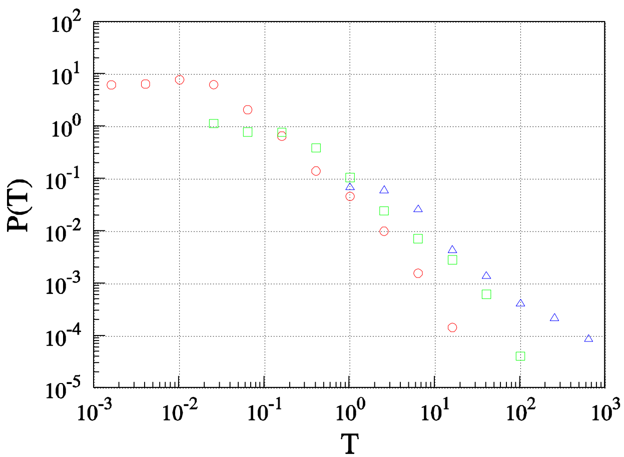
Numerical results



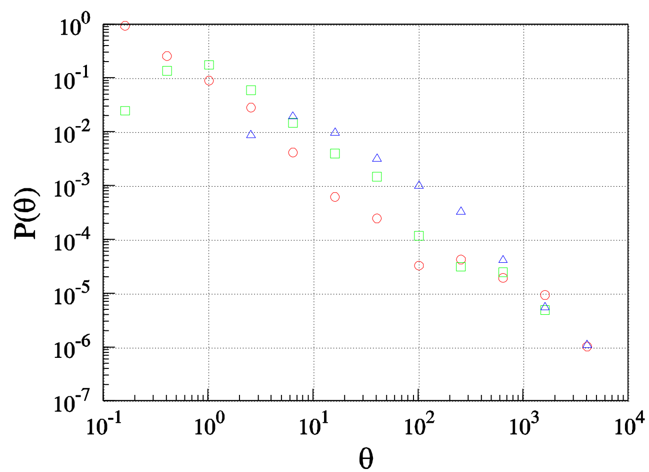
Burst size *S* vs burst duration *T*



Distribution of bust size *P*(*S*)



Distribution of bust durations *P*(*T*)



Distribution of interbust time *P*(*θ*)

Lamberti transformation $y(x) = \frac{1}{(\eta - 1)x^{\eta-1}}$

equation

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1}\mathrm{d}t_s + x^{\eta}\mathrm{d}W_s$$

convert to Bessel process

$$\mathrm{d}y = \left(\nu + \frac{1}{2}\right)\frac{\mathrm{d}t_s}{y} + \mathrm{d}W_s$$

i.e., N-dimensional Brownian diffusion with

$$N = 2(\nu + 1) = \frac{\lambda - 1}{\eta - 1}$$

Distribution of burst durations T. Theory

$$p_{h_y}^{(\nu)}(t) \approx C_2 \int_{j_{\nu,1}}^{\infty} x^2 \exp\left(-\frac{x^2 t}{2h_y^2}\right) \mathrm{d}x =$$

$$= C_2 \left[\frac{h_y^2 j_{\nu,1} \exp\left(-\frac{j_{\nu,1}^2 t}{2h_y^2}\right)}{t} + \sqrt{\frac{\pi}{2}} \frac{h_y^3 \operatorname{erfc}\left(\frac{j_{\nu,1}\sqrt{t}}{\sqrt{2}h_y}\right)}{t^{3/2}} \right]$$

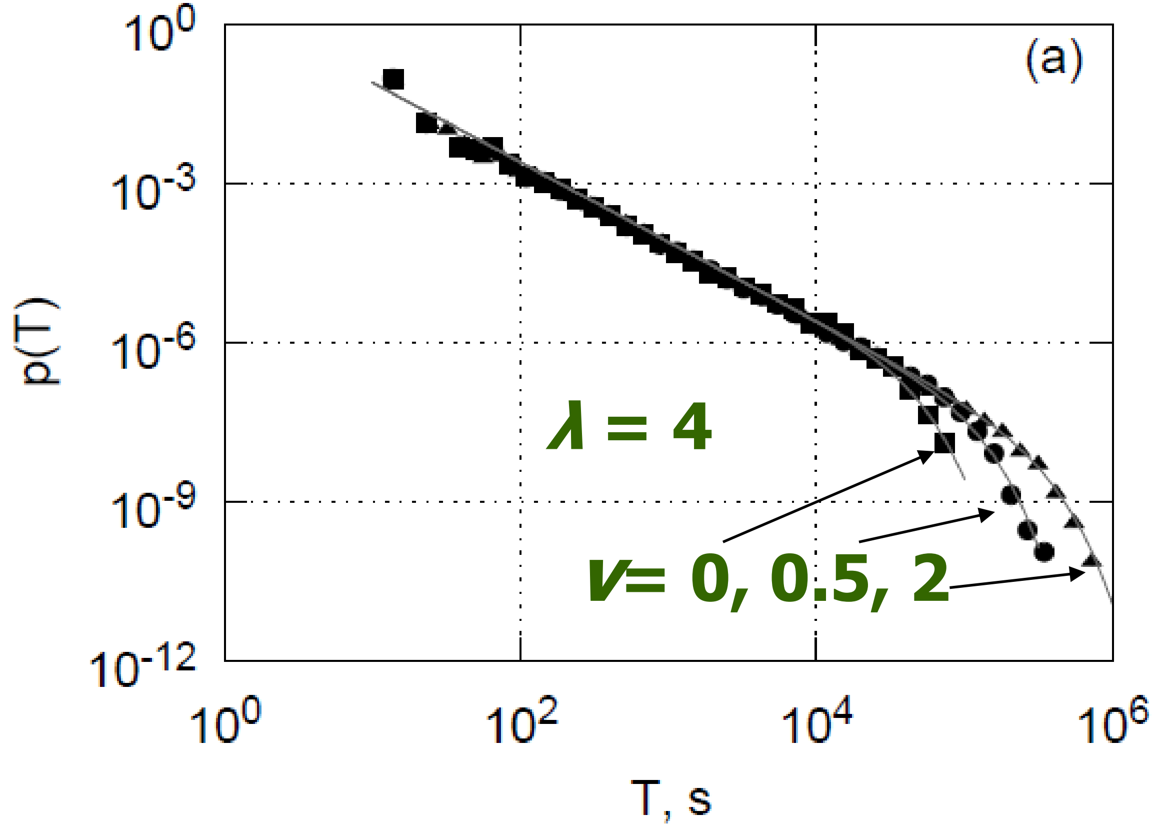
$$p_{h_y}^{(\nu)}(t) \sim t^{-3/2}, \quad \text{when} \quad t \ll \frac{2h_y^2}{j_{\nu,1}^2},$$

$$p_{h_y}^{(\nu)}(t) \sim \frac{\exp\left(-\frac{j_{\nu,1}^2 t}{2h_y^2}\right)}{t}, \quad \text{when} \quad t \gg \frac{2h_y^2}{j_{\nu,1}^2}$$

$j_{\nu,k}$ **is a k-th zero of Bessel function** J_ν

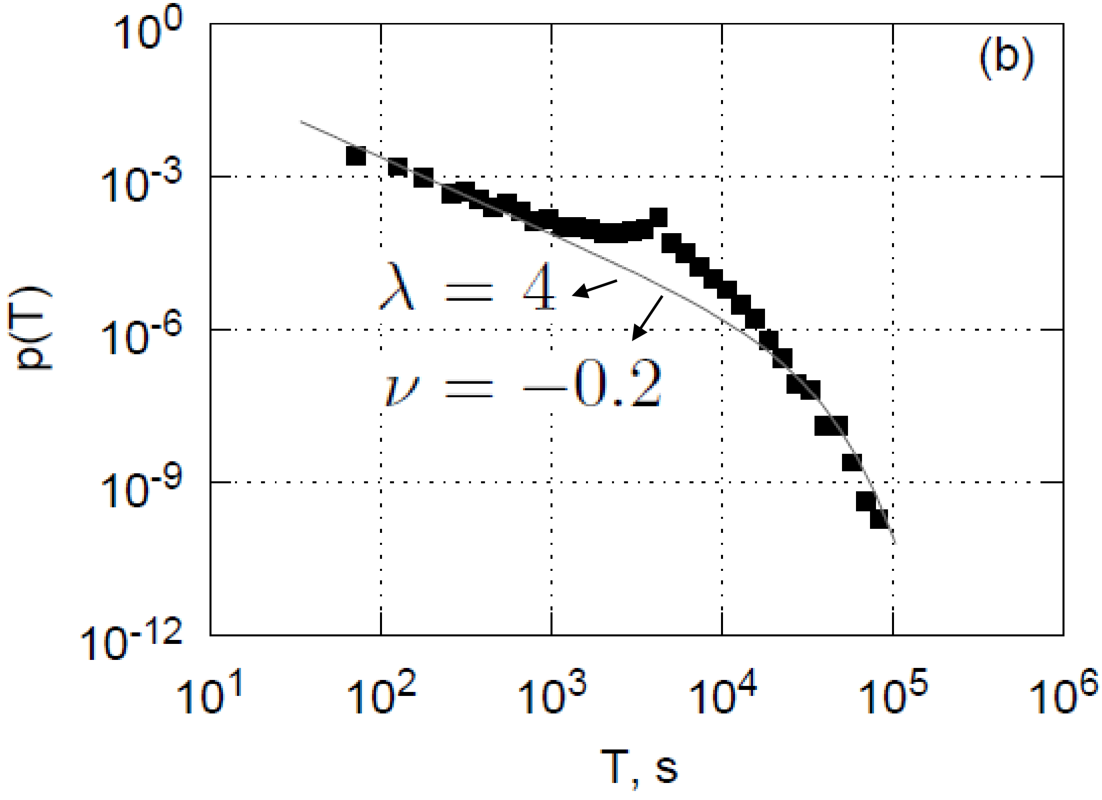
V. Gontis, A. Kononovicius and S. Reimann, ACS, arXiv:1201.3083v1

Distribution of burst durations *T*.
Comparison of analytical results with calculations



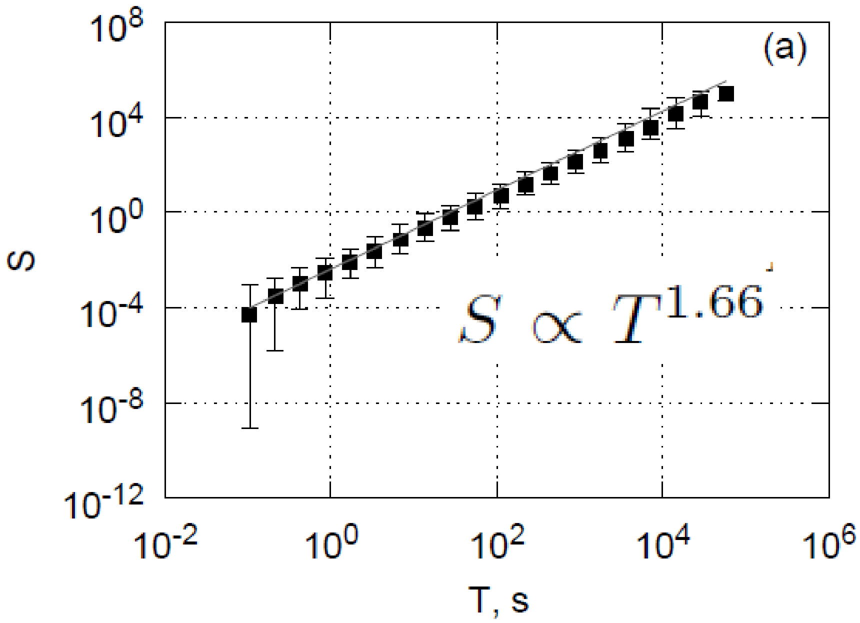
Distribution of burst durations *T*.

Comparison of analytical results with empirical data

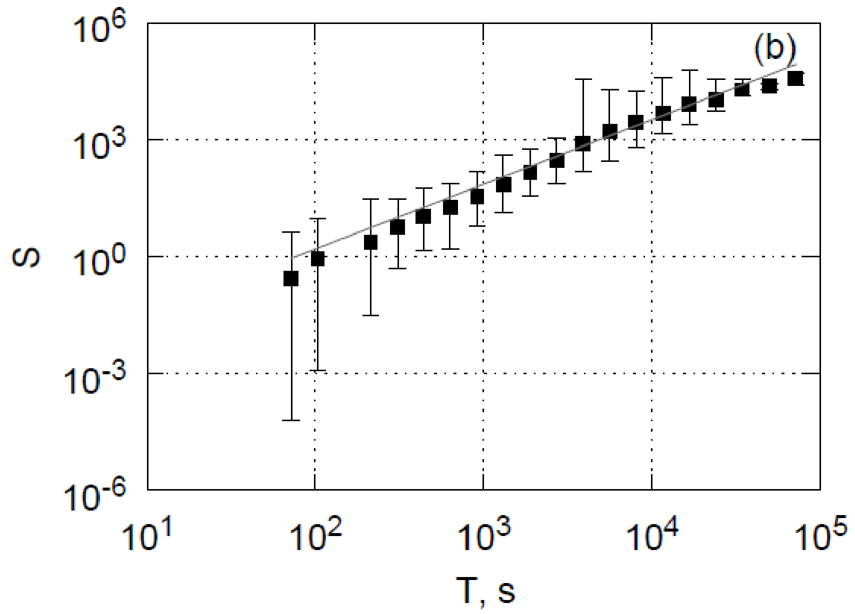


Comparison with empirical data for average of 24 stocks: ABT, ADM, BMY, C, CVX, DOW, FNM, GE, GM, HD, IBM, JNJ, JPM, KO, LLY, MMM, MO, MOT, MRK, SLE, PFE, T, WMT, XOM.

Burst size *S* vs burst duration *T*

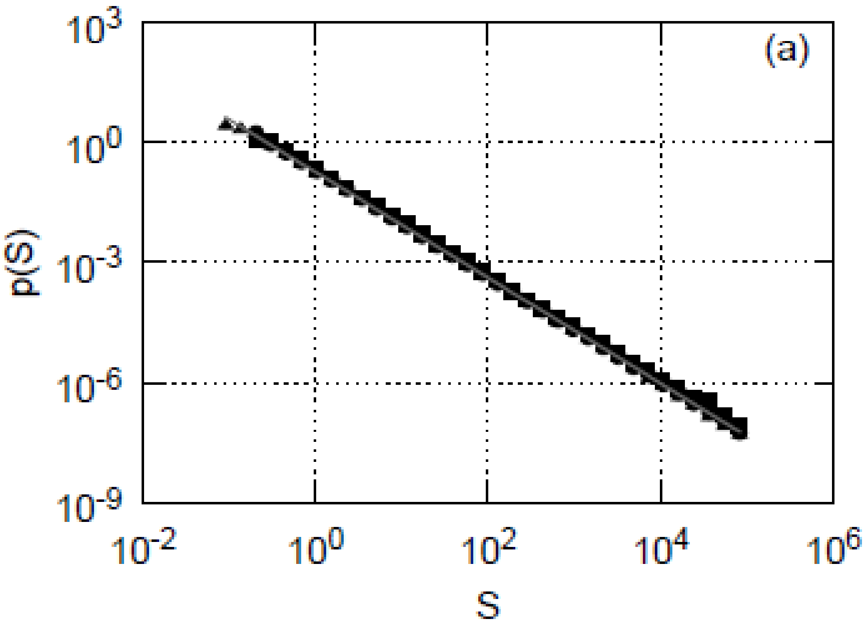


Theoretical results

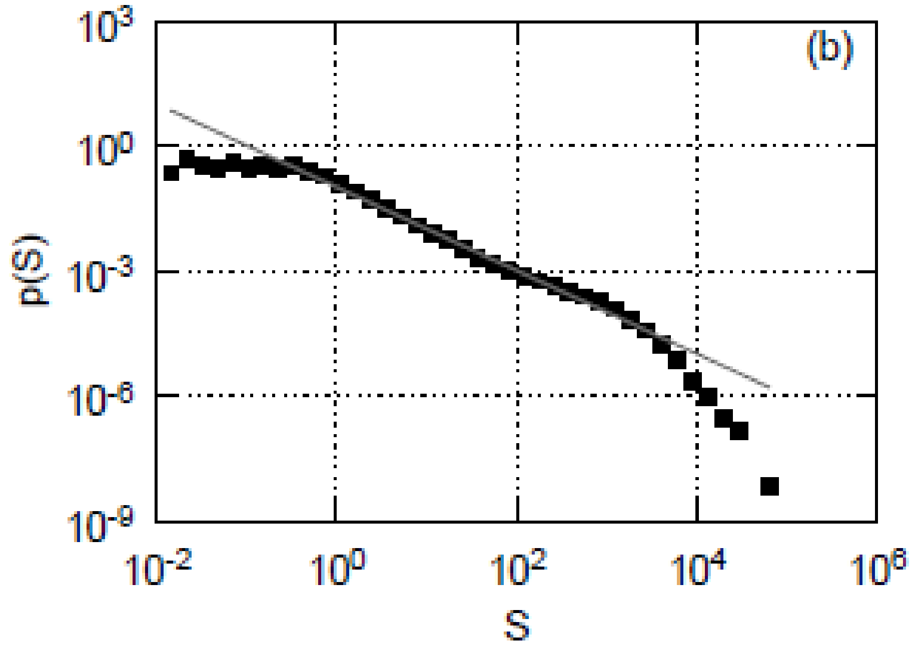


Empirical data

PDF of the bursts size



Theoretical results



Empirical data

Some conclusions

- Nonlinear stochastic differential equation
- may generate the inverse cubic
- q-exponential and
- q-Gaussian distributed signals with
- $1/f^{\beta}$ power spectrum,
- exhibiting bursts, similar to observable in empirical data.