1/f Fluctuations from the Microscopic Herding Model

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with

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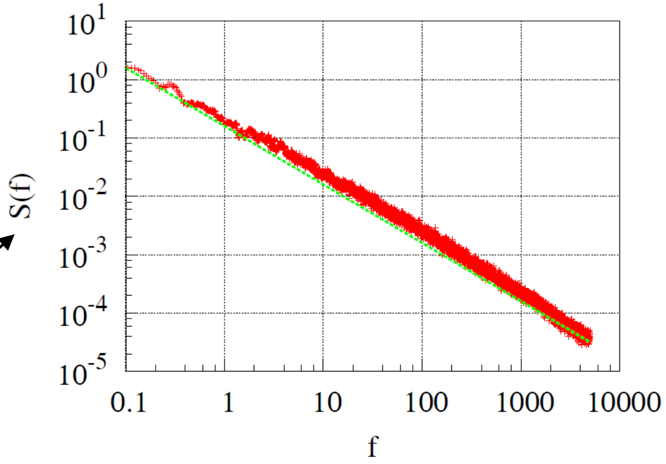
Focus of the talk

Our researches are related with the

- **1/f noise problem**
- **Nonlinear stochastic differential equations and**
- **Herding processes**

1/*f* (One-Over-F) Noise or 1/*f* Fluctuations

- **1/*f* noise, occasionally called**
- **"flicker noise" or "pink noise"**
- **is a type of noise whose**
- **power spectral density**
- ***S* (*f*) as a function of**
- **the frequency *f***
- **behaves like *S* (*f*) ~ *1/f* $^{\beta}$**
- **where the exponent β =1 or is close to 1.**



1/*f* (One-Over-F) Noise or 1/*f* Fluctuations

- Since the first observation of 1/*f* noise
- by Johnson in 1925,
- fluctuations of signals exhibiting 1/*f* behavior
- of the power spectral density at low frequencies
- have been observed in a wide variety of physical, geophysical, biological,
- financial, traffic, Internet, astrophysical and other systems

Puzzles, mystery of 1/*f* noise

- **1/*f* noise is intermediate between white noise: no correlation in time, *S* (*f*) ~1/*f* ^{0},**

$$I(t)=\sigma\xi(t),\quad \langle\xi(t)\xi(t')\rangle=\delta(t-t')$$

- **and Brownian motion: no correlation between increments, *S* (*f*) ~1/*f* ^{2},**

$$\mathrm{d}I=\sigma\mathrm{d}W(t),\quad \int_0^t\xi(t')\,\mathrm{d}t'=W(t)$$

$W(t)$ is Wiener process (Brownian motion)

Puzzles, mystery of 1/ *f* noise

- **In contrast to the Brownian motion generated by the linear stochastic equation**
- **Simple systems of linear stochastic differential equations, generating signals with 1/ *f* noise are not known**

These results make the problem of the omnipresent 1/ f noise

one of the oldest puzzles

in contemporary physics

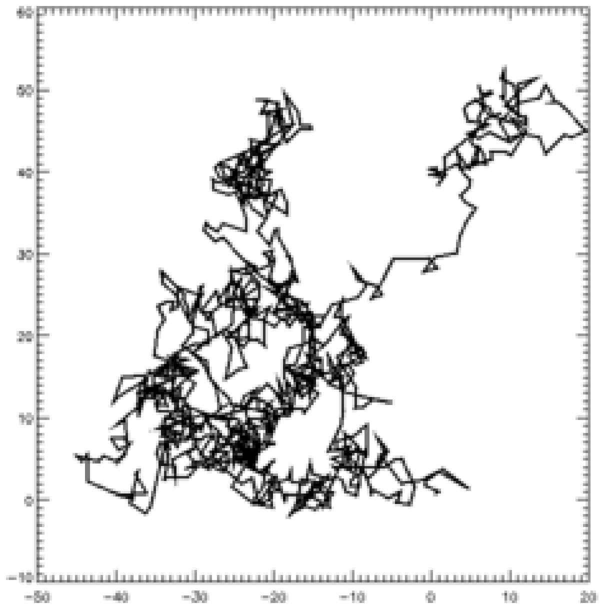
Historical Remarks.
Brownian motion (1)

1. Robert Brown (1827)

"…Microscopical observation of active molecules…"

Brownian motion

in space



B. Kaulakys, Vilnius University, Lithuania: **www.itpa.lt/kaulakys**

Brownian motion (2)

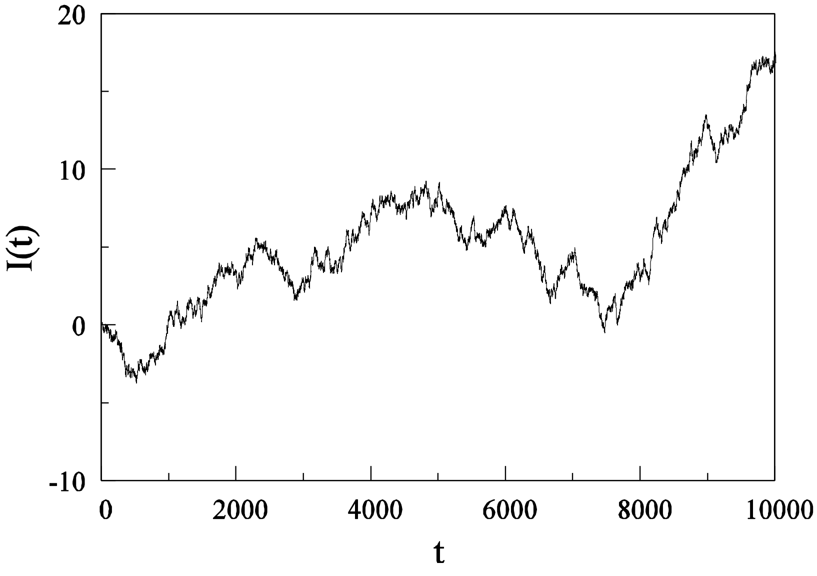
2. Louis Bachelier (1900)

"Théorie de la spéculation"

A theory of Brownian motion, Pioneering Econophysics

Brownian motion

of the intensity of the signal



Brownian motion (3)

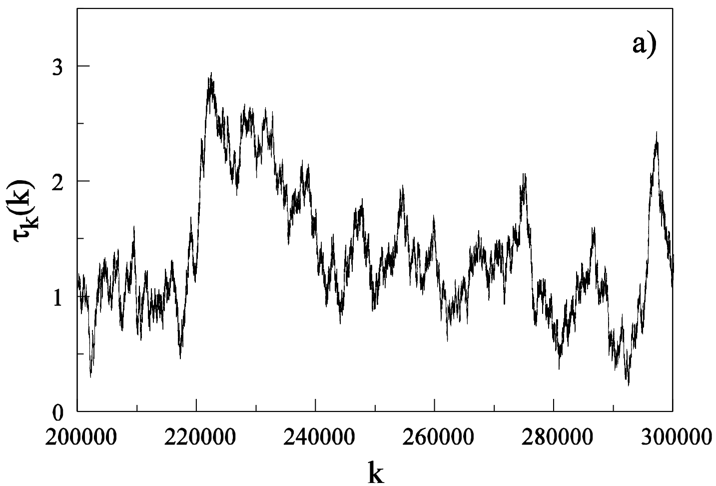
B.K., T.Meskauskas, V.Gontis, J.Ruseckas and M.Alaburda models (1997-2011)

Brownian or Brownian-like motion

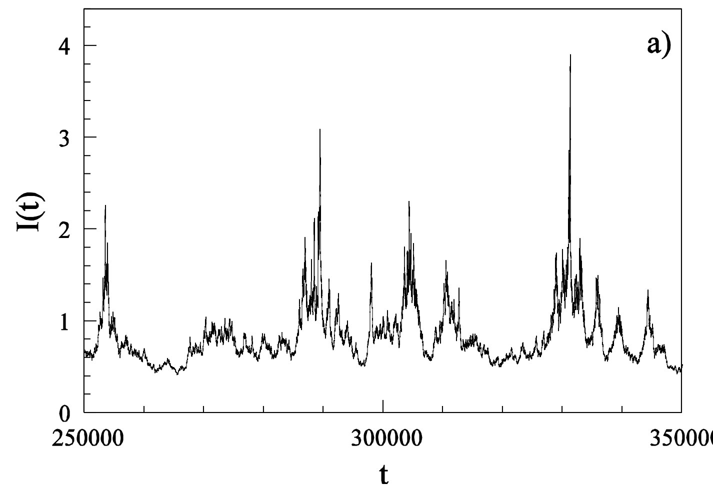
in time axis

(of the mean inter-event, inter-pulse time)

as one of possible origins of 1/*f* noise



Motion of the interevent time



The signal

POINT PROCESS MODEL OF 1/f NOISE

The signal of the model consists of pulses or events

$$I(t)=\sum_k A_k(t-t_k)$$

In a low frequency region and for long-range correlations we can restrict analysis to the noise originated from the correlations between the occurrence times t_k.

Therefore, we can simplify the signal to the point process

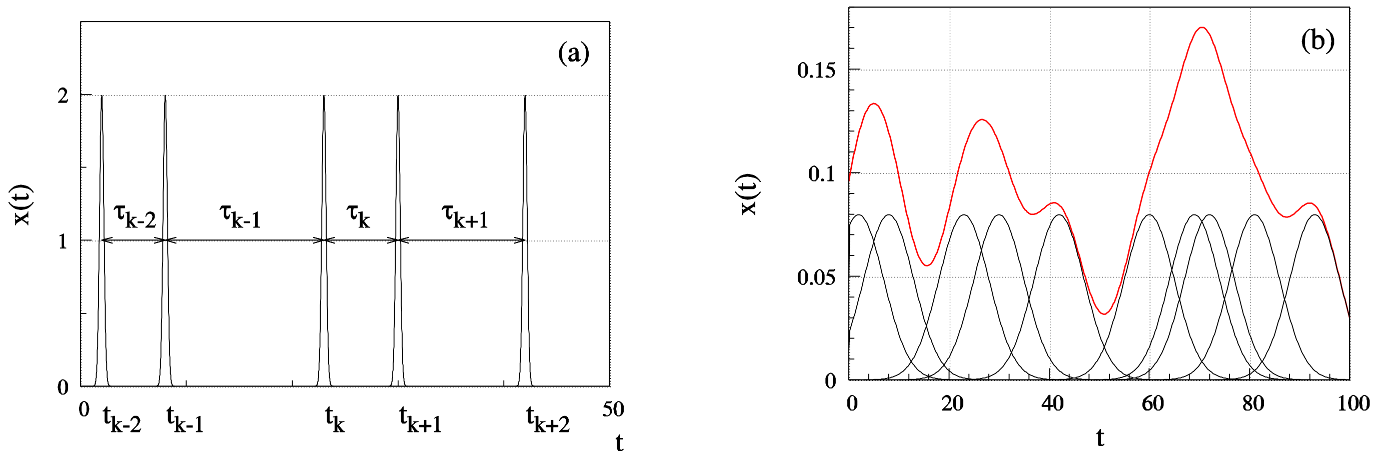
The point process

$$I(t)=\bar{a}\sum_k \delta(t-t_k)$$

is primarily and basically defined by the occurrence times t_1, t_2, …t_k…

Or by the interevents times $\tau_k=t_{k+1}-t_k$

Power spectral density of the point process



$$S(f) = \lim_{T\to\infty}\left\langle \frac{2}{T}\int_{t_i}^{t_f}\int_{t_i}^{t_f} I(t')\, I\left(t''\right) e^{i\omega\left(t''-t'\right)}\mathrm{d}t'\mathrm{d}t''\right\rangle$$

$$= \lim_{T\to\infty}\left\langle \frac{2\bar{a}^2}{T}\sum_{k=k_{\min}}^{k_{\max}}\ \sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i\omega\Delta(k;q)}\right\rangle$$

where $T = t_f - t_i \gg \omega^{-1}$ is the observation time, $\omega = 2\pi f$, and

may be calculated directly

$$\Delta(k;q) \equiv t_{k+q} - t_k = \sum_{i=k}^{k+q-1} \tau_i$$

Stochastic multiplicative point process

Quite generally the dependence of the mean interpulse time on the occurrence number *k* may be described by the general Langevin equation with the drift coefficient $d(\tau_k)$

and a multiplicative noise $b(\tau_k)\xi(k)$

$$\frac{d\tau_k}{dk} = d(\tau_k) + b(\tau_k)\,\xi(k)\,.$$

Replacing the averaging over *k* by the averaging over the distribution of the interpulse times τ_k, $P_k(\tau_k)$, **we have the power spectrum**

$$S(f) = 4\bar{I}^2\bar{\tau}\int_0^\infty d\tau_k P_k(\tau_k)\,\mathrm{Re}\int_0^\infty dq\exp\left\{i\omega\left[\tau_k q + d(\tau_k)\frac{q^2}{2}\right]\right\}$$

$$= 2\bar{I}^2\frac{\bar{\tau}}{\sqrt{\pi}f}\int_0^\infty P_k(\tau_k)\,\mathrm{Re}\left[e^{-i\left(x-\frac{\pi}{4}\right)}\operatorname{erfc}\sqrt{-ix}\right]\frac{\sqrt{x}}{\tau_k}d\tau_k$$

✓**B. K., V. Gontis, M. Alaburda, Phys. Rev. E 71,** 051105 **(2005)**

Multiplicative point process

Iterative equation for the mean interevent time

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma \tau_k^{\mu} \varepsilon_k .$$

$$P_k(\tau_k) = \frac{1+\alpha}{\tau_{\max}^{1+\alpha} - \tau_{\min}^{1+\alpha}} \tau_k^{\alpha}, \quad \alpha = \frac{2\gamma}{\sigma^2} - 2\mu. \quad \beta = 1 + \frac{\alpha}{3-2\mu}, \quad \frac{1}{2} < \beta < 2.$$

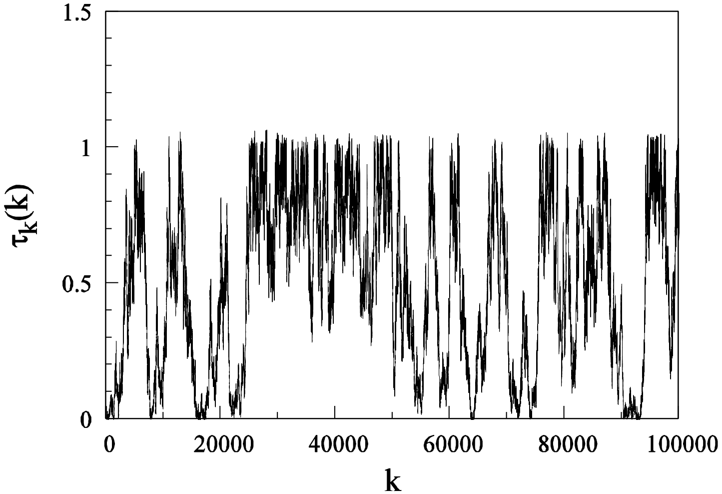
Power spectral density

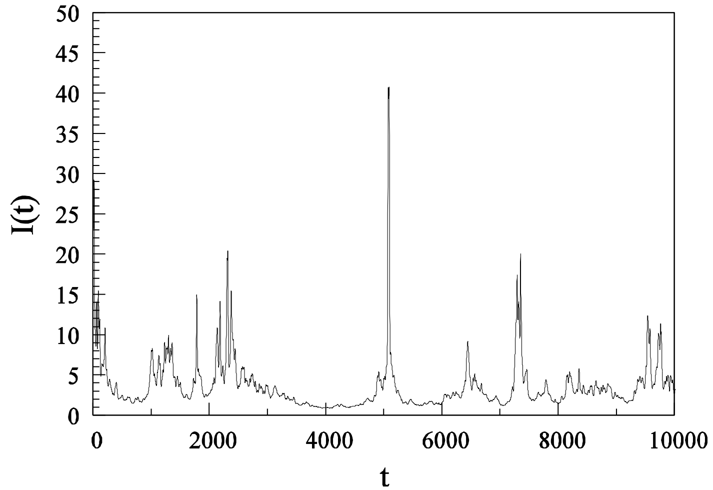
$$S(f) = \frac{(2+\alpha)(\beta-1)\bar{a}^2 \Gamma(\beta - 1/2)}{\sqrt{\pi}\alpha \left(\tau_{\max}^{2+\alpha} - \tau_{\min}^{2+\alpha}\right) \sin(\pi \beta/2)} \left(\frac{\gamma}{\pi}\right)^{\beta-1} \frac{1}{f^{\beta}}$$

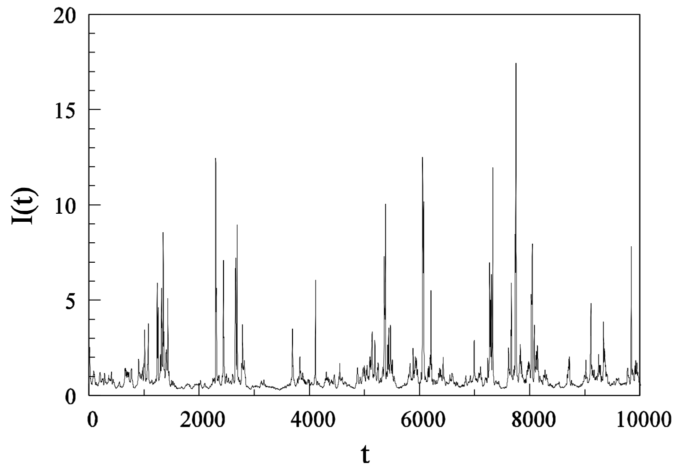
Distribution density of the signal intensity $I \,\square\, \bar{a}/\tau_k$ **is**

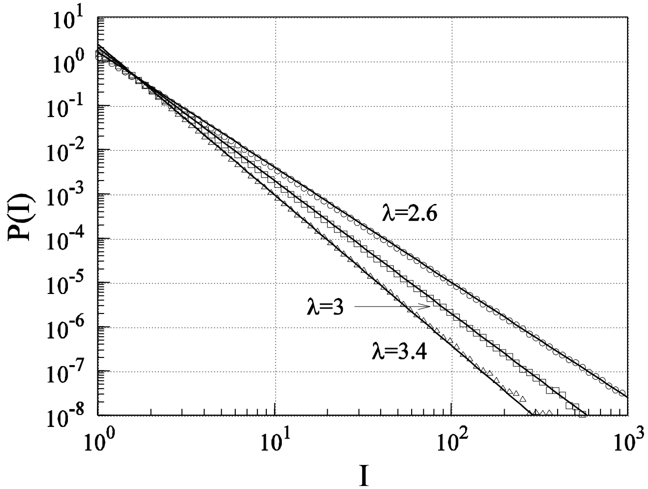
$$P(I) = \frac{\bar{a}\bar{I}}{I^3} P_k\left(\frac{\bar{a}}{I}\right). \qquad P(I) = \frac{\lambda - 1}{\tau_{\max}^{\lambda-1} - \tau_{\min}^{\lambda-1}} \frac{\bar{a}^{\lambda-1}}{I^{\lambda}}, \; \lambda = 3 + \alpha.$$

Signal of the point process. Simulated examples

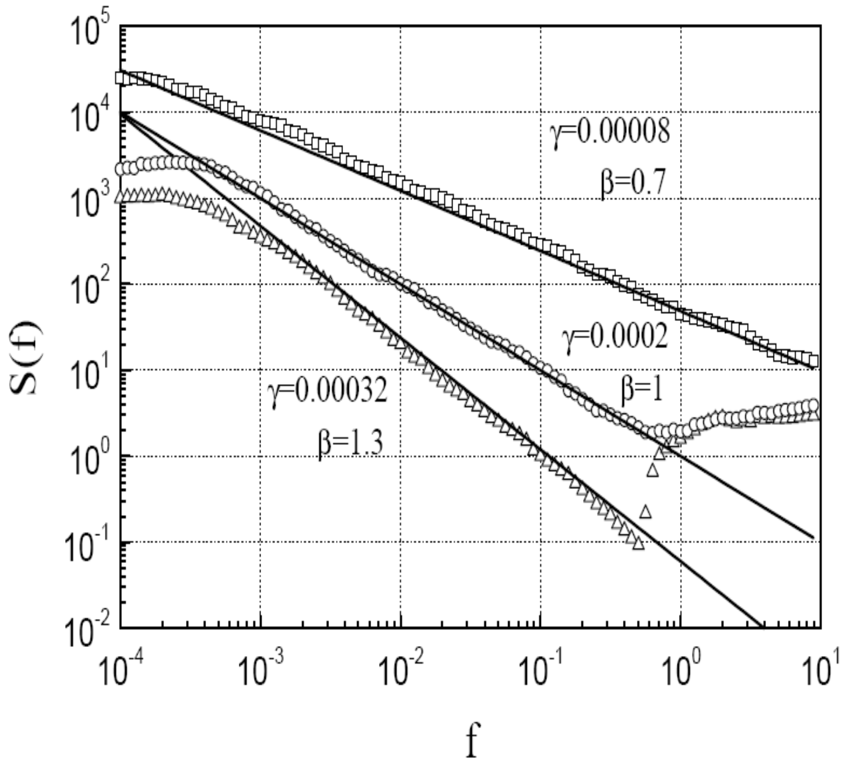


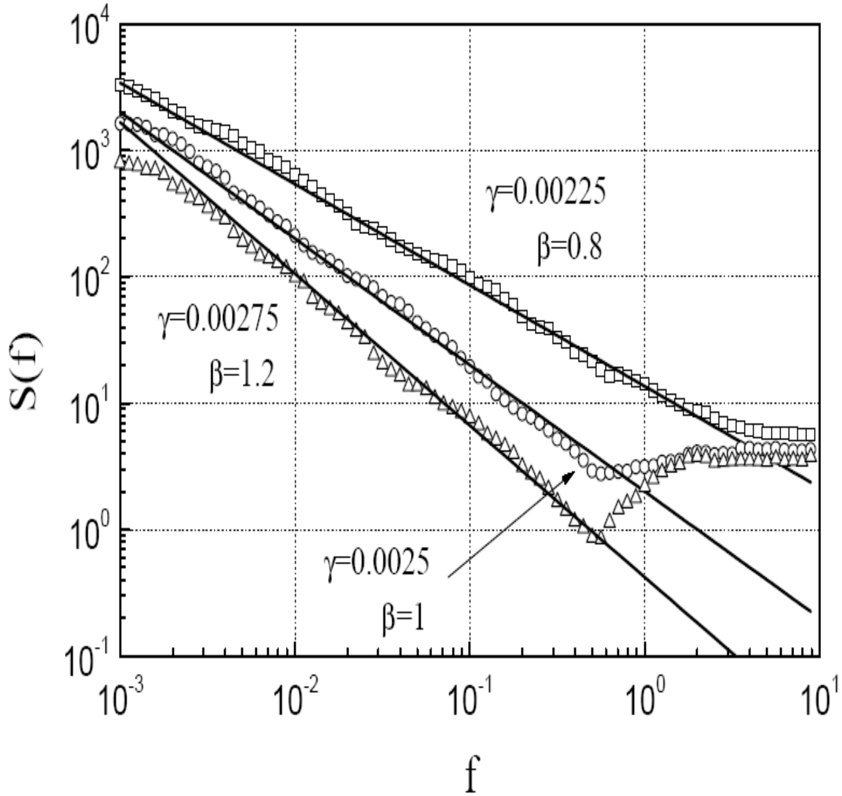






Power spectrum *S*(*f*)





Averaged over 10 realizations of pulse 10^6 sequences with the parameters

$\bar{a}=1,\ \mu=\frac{1}{2},\ \sigma=0.02$ **and different γ**

For $\bar{a}=1,\ \mu=1,\ \sigma=0.05$

and different γ

Summarizes of our point process models

- **We have presented simple point process models of $1/f^{\beta}$ noise, covering different values of the exponent β.**
- **The proposed models relates and connects the power-law spectral density with the power-law distribution of the signal intensity into the consistent theoretical approach.**

Nonlinear stochastic differential equation (SDE) generating 1/f noise from the point process model

$$\tau_{k+1} = \tau_k + \sigma\varepsilon_k\,, S(f) \propto 1/f$$

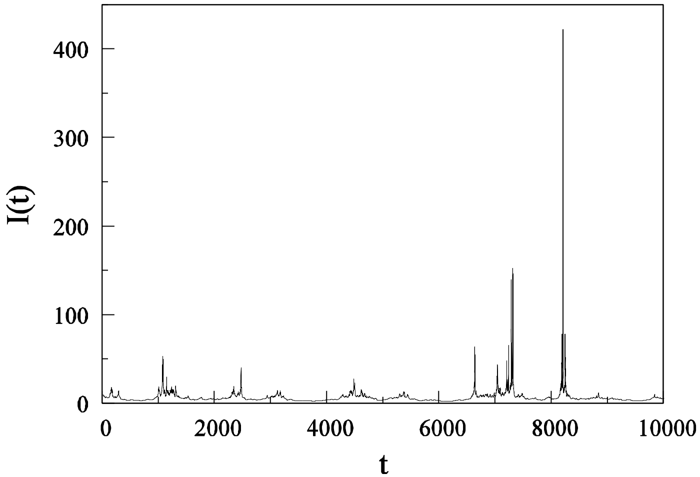
$$\frac{d\tau_k}{dk} = \sigma\xi(k) \quad \langle \xi(k)\xi(k')\rangle = \delta(k-k')$$

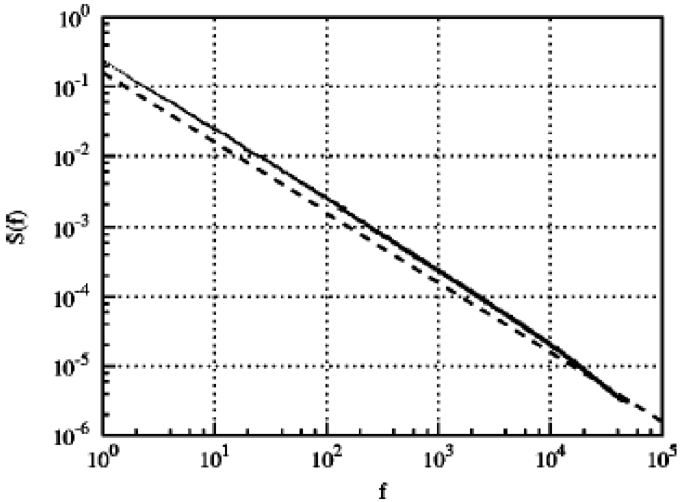
$$\mathrm{d}t = \tau_k \mathrm{d}k, \qquad x = a/\tau_k$$

$$\frac{dx}{dt} = x^4 + x^{5/2}\xi(t), \; S(f) \propto 1/f$$

$$P(x) \sim \frac{1}{x^3}$$

1/*f* noise and power-law distribution





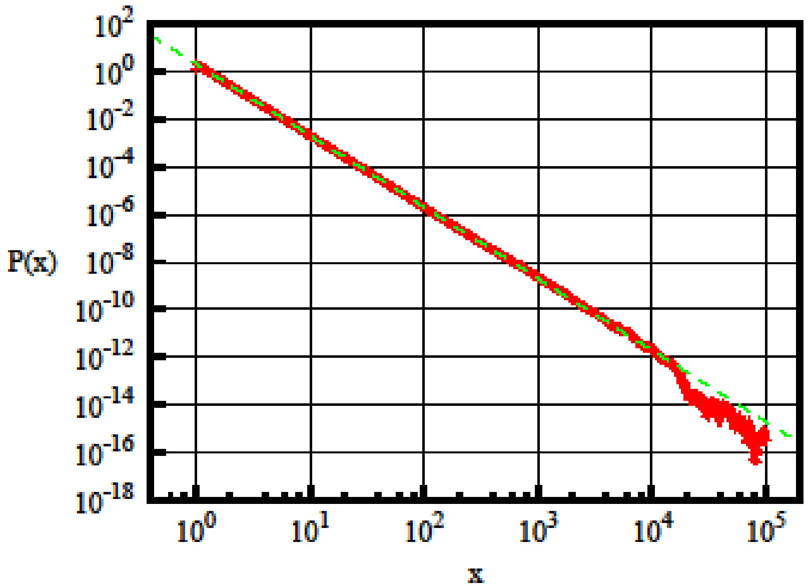
✓**B. K. and J. Ruseckas, Phys. Rev. E 70, 020101(R) (2004)**

Simplest nonlinear stochastic differential equations (SDE) generating signals with $1/f^{\beta}$ fluctuations

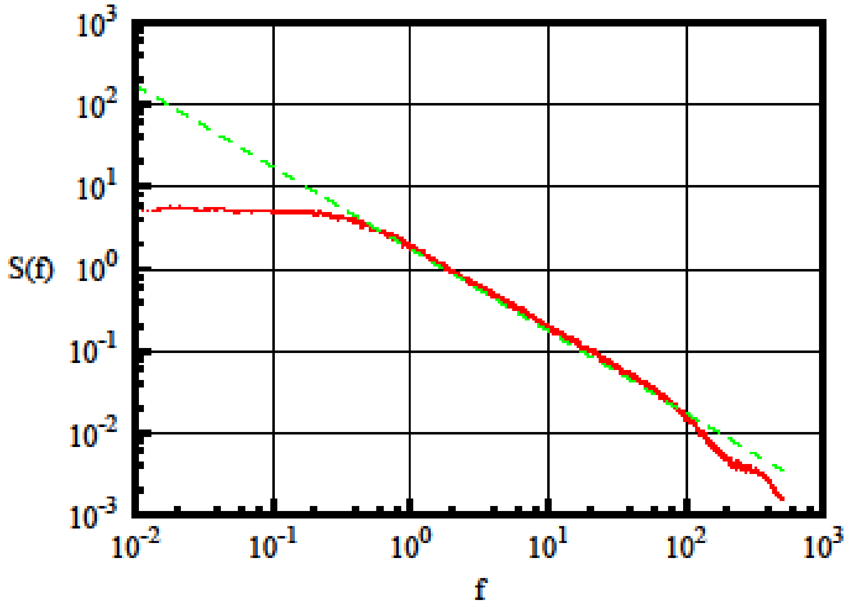
a) equation

$$dx = x^{3/2} dW, \quad \eta = \frac{3}{2}, \quad \lambda = 3, \quad \beta = 1$$

in Ito convention



Signal distribution



Power spectrum

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Other simple nonlinear stochastic differential equations (SDE) generating signals with $1/f^{\beta}$ fluctuations

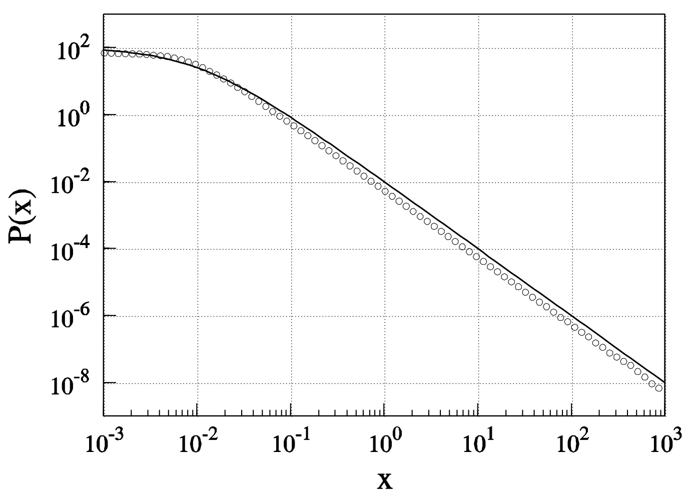
b) equation

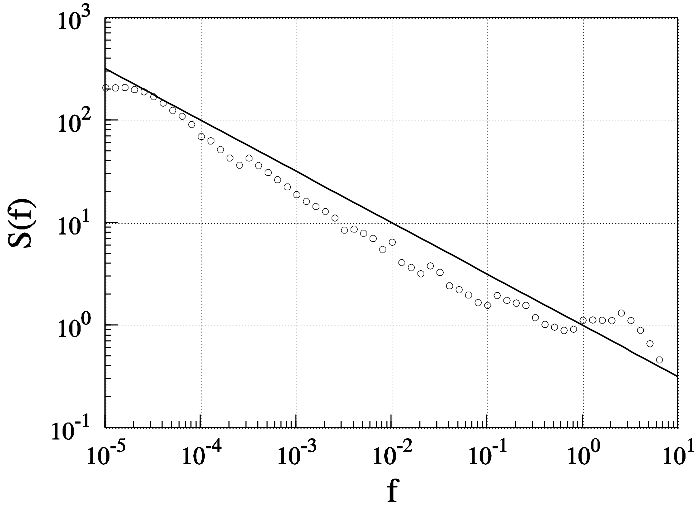
$$\mathrm{d}x = x^2 \circ \mathrm{d}W, \quad \eta = 2, \quad \lambda = 2, \quad \beta = \frac{1}{2}$$

or

$$\mathrm{d}x = \left(x_m^2 + x^2\right) \circ \mathrm{d}W, \quad \eta = 2, \quad \lambda = 2, \quad \beta = \frac{1}{2}$$

in Stratonovich convention





Other simple (SDE) generating signals with $1/f^{\beta}$ fluctuations

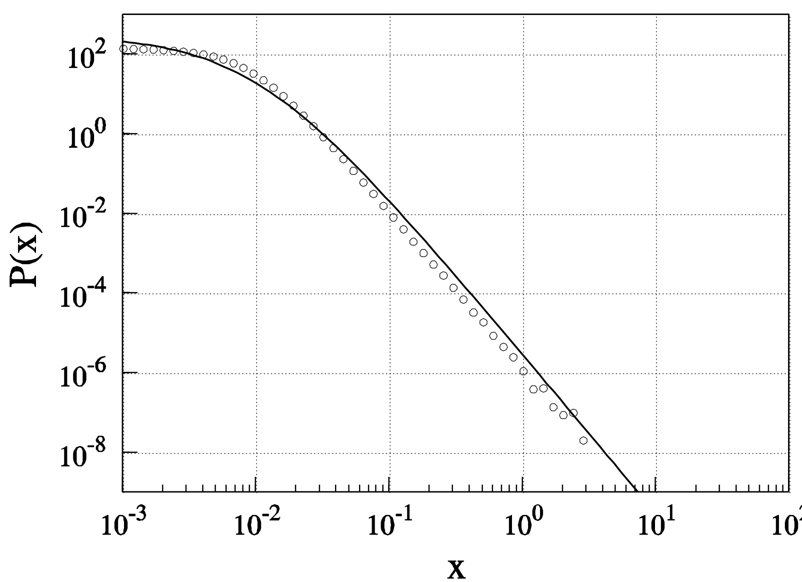
c) equation

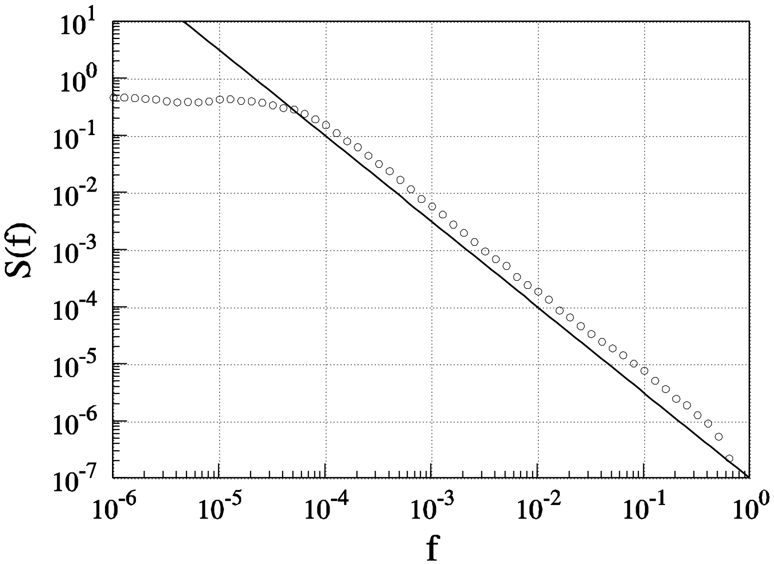
$$dx = x^2 dW, \quad \eta = 2, \quad \lambda = 4, \quad \beta = \frac{3}{2}$$

or

$$dx = \left(x_m^2 + x^2\right) dW, \quad \eta = 2, \quad \lambda = 4, \quad \beta = \frac{3}{2}$$

in Ito convention





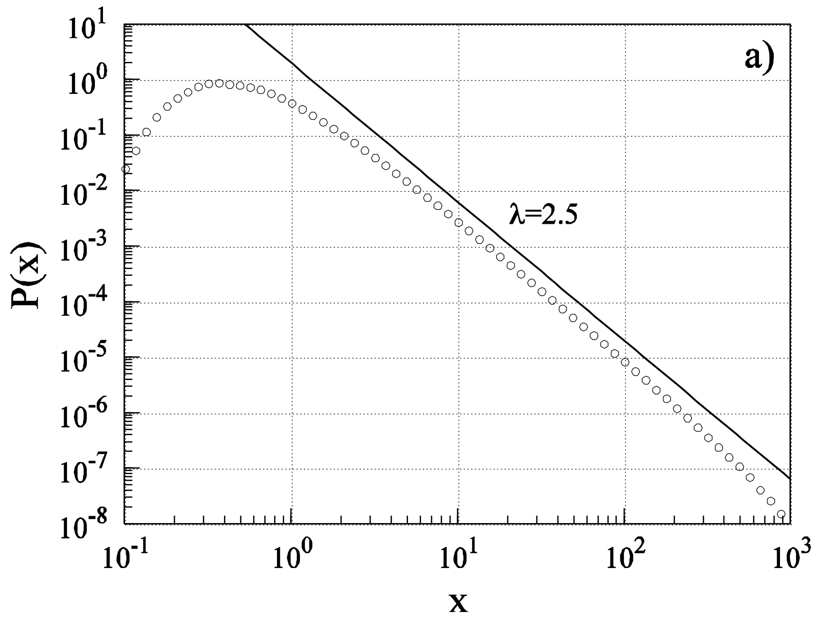
Generalisation for $1/f^{\beta}$ noise

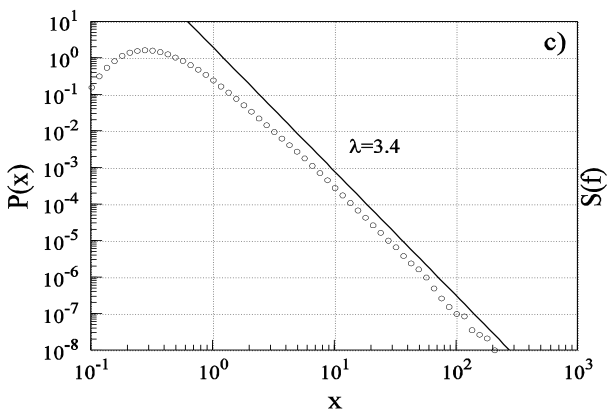
$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma \tau_k^{\mu} \varepsilon_k .$$

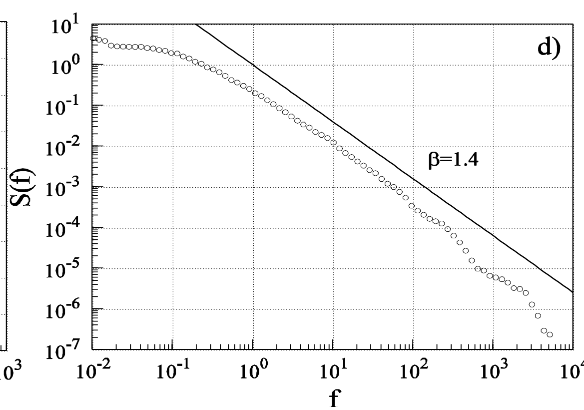
$$\frac{dx}{dt_s} = \Gamma x^{2\eta-1} + x^{\eta} \xi(t_s)$$

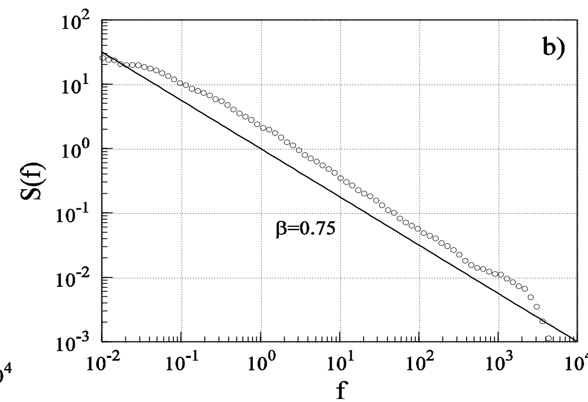
$$P(x) \sim \frac{1}{x^{\lambda}}, \quad \lambda = 2(\eta - \Gamma)$$

$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 2 - \frac{2\Gamma + 1}{2\eta - 2}.$$









✓**B. K. et al, Physica A 365,** 217 **(2006)**

Simples nonlinear (SDE)
generating $1/f^{\beta}$ noise and $P(x)\sim 1/x^{\lambda}$ distribution

$$\mathrm{d}x = \Gamma x^3 \mathrm{d}t + x^2 \mathrm{d}W$$

with $\beta = \frac{3}{2} - \Gamma$ **and** $\lambda = 4 - 2\Gamma$

Another form and improvement of the equation

$$\mathrm{d}x = \left(2 - \frac{1}{2}\lambda\right)(x_m + x)^3 \mathrm{d}t + (x_m + x)^2 \mathrm{d}W$$

where $\Gamma = 2 - \frac{1}{2}\lambda$

Normalized distribution of the signal

$$P(x) = \frac{(\lambda - 1)\, x_m^{\lambda-1}}{(x_m + x)^{\lambda}}, \quad x > 0.$$

Without the divergence

q-exponential distribution

$$dx = \left(\eta - \frac{1}{2}\lambda\right)(x_m + x)^{2\eta-1}\,dt + (x_m + x)^{\eta}\,dW$$

(i) is linear for small x << x_m ,

(ii) restrict divergence of power-law distribution of x at x=0

$$P(x) = \frac{(\lambda-1)x_m^{\lambda-1}}{(x_m+x)^{\lambda}}$$

$$= \frac{(\lambda-1)}{x_m}\exp_q\left\{-\lambda\frac{x}{x_m}\right\}, \quad x > 0$$

q-exponent

and

(iii) generate signals with 1/f $^{\beta}$ spectrum:

Analytical calculations from the related point process model

$$S(f) \approx \frac{A}{f^{\beta}}, \quad \frac{1}{2} < \beta < 2, \quad 4-\eta < \lambda < 1+2\eta,$$

$$\beta = 1 + \frac{\lambda-3}{2(\eta-1)}, \quad \eta > 1,$$

$$A \approx \frac{(\lambda-1)\,\Gamma(\beta-1/2)\,x_m^{\lambda-1}}{2\sqrt{\pi}(\eta-1)\sin(\pi\beta/2)}\left(\frac{2+\lambda-2\eta}{2\pi}\right)^{\beta-1}$$

- ✓ **B. K. and M. Alaburda, J. Stat. Mech. P02051 (2009)**

Autocorrelation of the signal with $1/f^{\beta}$ noise

$$C(s) = \langle x(t)\, x(t+s) \rangle = \int_0^{\infty} S(f) \cos(2\pi f s)\, \mathrm{d}f$$

Power spectral density may be approximated as

$$S(f) = \frac{A}{\left(f_0^2 + f^2\right)^{\beta/2}} \Rightarrow \begin{cases} A/f_0^{\beta}, & f \to 0, \\ A/f^{\beta}, & f \gg f_0 \end{cases}$$

Autocorrelation may be expressed via the modified Bessel functions $K_{\nu}(z)$

$$C(s) = \frac{A\sqrt{\pi}}{\Gamma(h+1/2)} \left(\frac{\pi s}{f_0}\right)^h K_{|h|}(2\pi f_0 s) = \frac{A}{f_0^{2h}} \frac{\sqrt{\pi}}{\Gamma(h+1/2)} \left(\frac{z}{2}\right)^h K_{|h|}(z)$$

where

$$h = \frac{\beta - 1}{2}$$

For $1 < \beta < 3$, h coincide with the Hurst exponent H

$$H \simeq \begin{cases} 0, & \beta < 1 \\ \frac{\beta-1}{2}, & 1 < \beta < 3 \\ 1, & \beta > 3 \end{cases}$$

Analytical expressions for leading terms of autocorrelations

(i) **β<1**

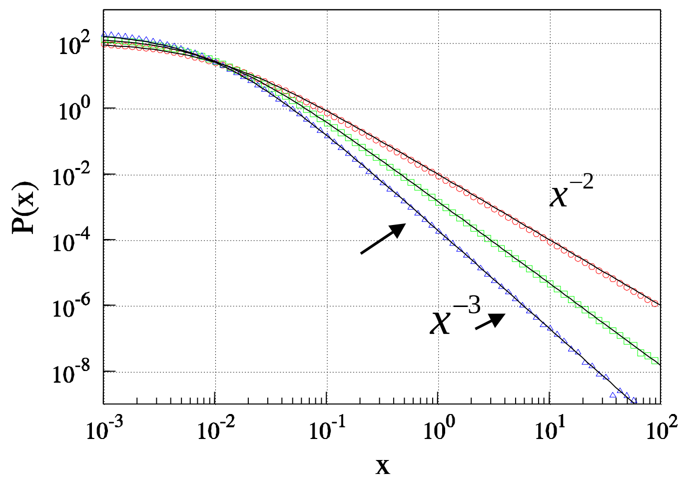
$$C(s) = \frac{A\sqrt{\pi}\Gamma\left(\frac{1-\beta}{2}\right)}{2\Gamma\left(\frac{\beta}{2}\right)} \frac{1}{(\pi s)^{1-\beta}} \sim \frac{1}{s^{1-\beta}}, \quad 0 < \beta < 1$$

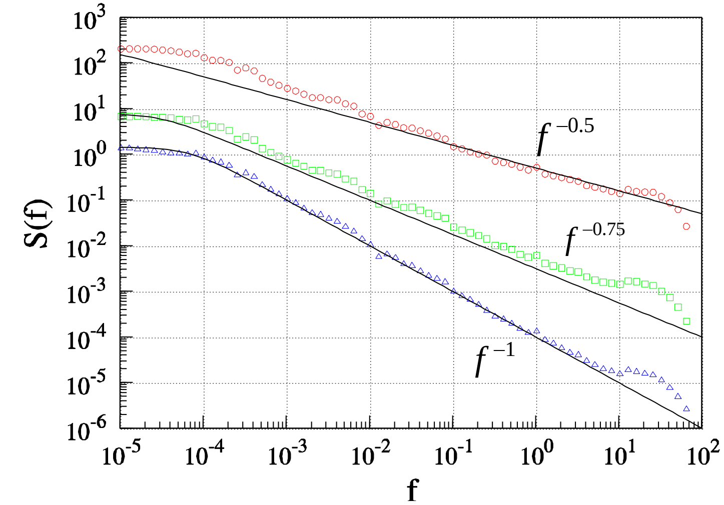
(ii) **β=1**

$$C(s) = AK_0\left(2\pi f_0 s\right) \simeq -A\left[\ln\left(2\pi f_{\min} s\right) + C\right], \quad C = 0.5772$$

$$C(s) \sim C_0 - A\ln s$$

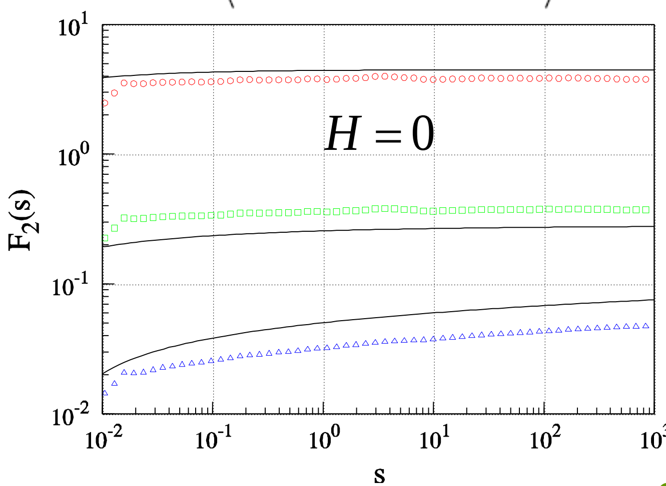
$$\mathrm{d}x = \left(2 - \frac{1}{2}\lambda\right)(x_m + x)^3\,\mathrm{d}t + (x_m + x)^2\,\mathrm{d}W$$ **, λ=2; 2.5 and 3**

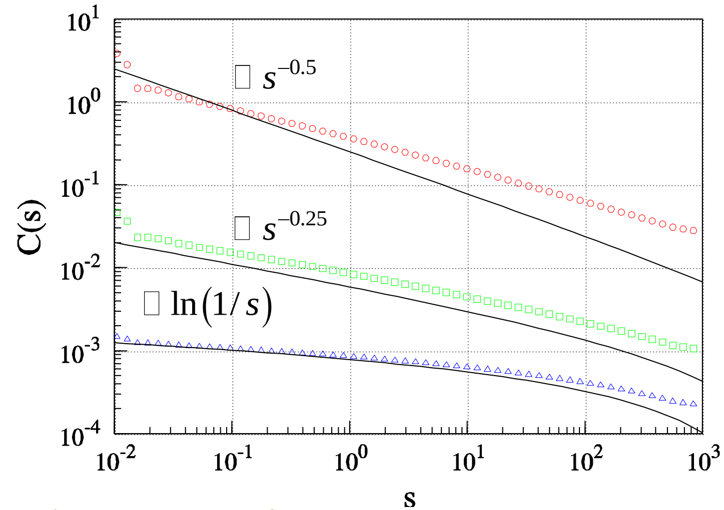




height-height correlation function (GHCF)

$$F_2(s) = \left\langle |x(t+s) - x(t)|^2 \right\rangle^{1/2}$$





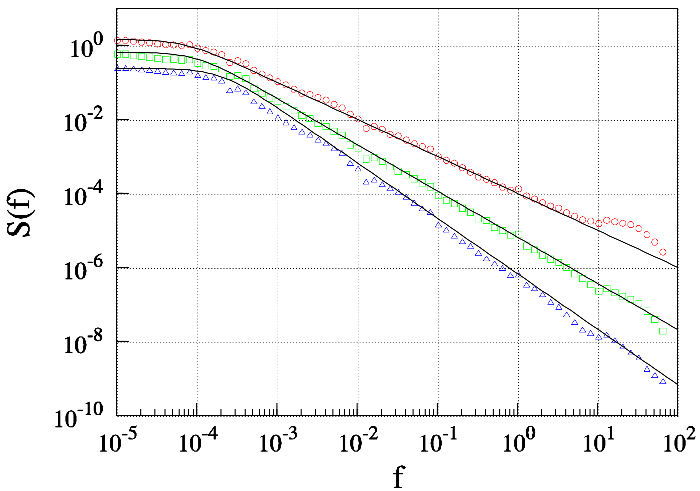
Analytical expressions for leading terms of autocorrelations

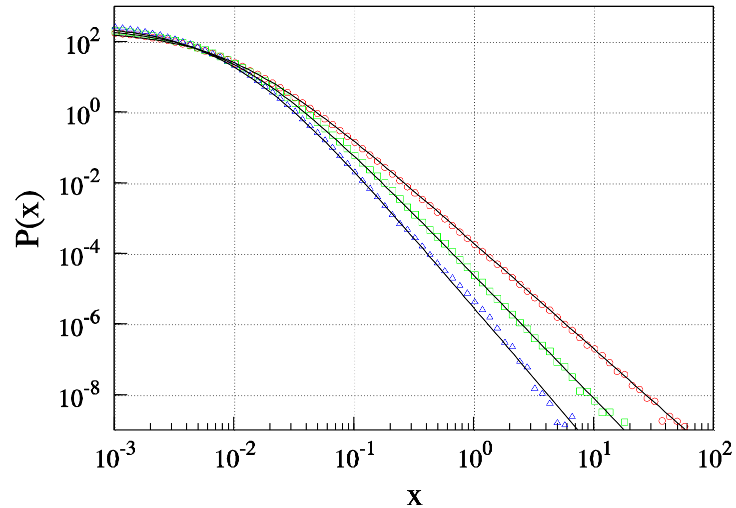
(iii) **β>1**

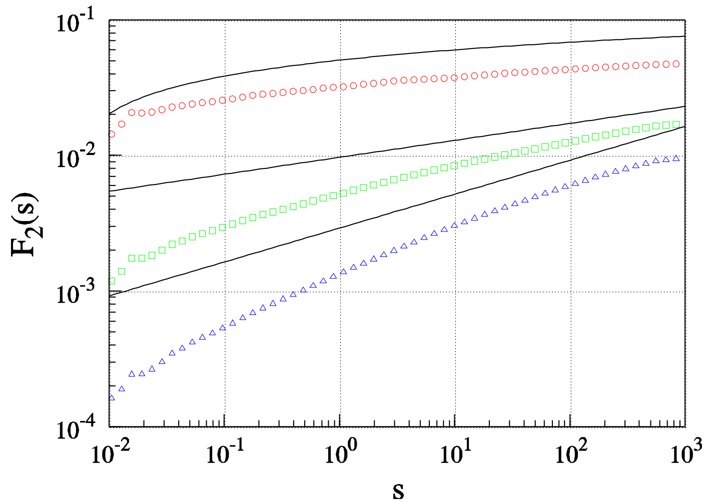
$$C(s) = C(0) - Bs^{\beta-1}, \quad 1 < \beta < 3,$$

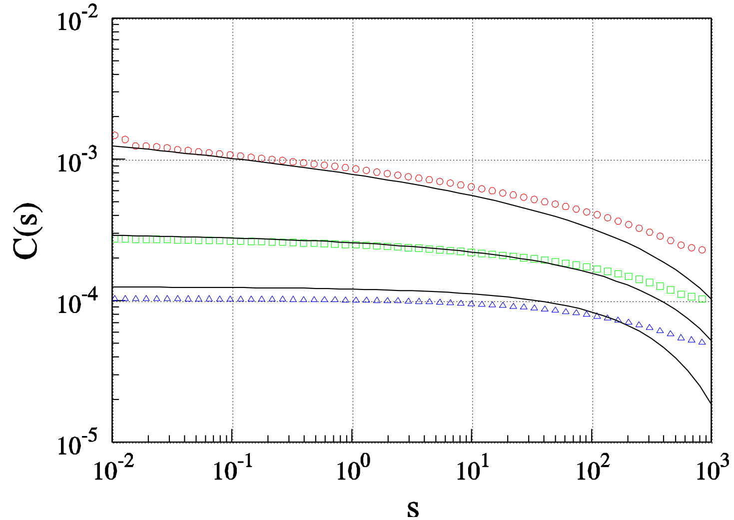
$$B = \frac{(2\pi)^{\beta-1}\,\Gamma(2-\beta)\sin(\pi\beta/2)}{(\beta-1)}A = -\frac{(2\pi)^{\beta}\,A}{4\Gamma(\beta)\cos(\pi\beta/2)}.$$

$$dx = \left(2 - \frac{1}{2}\lambda\right)(x_m + x)^3\, dt + (x_m + x)^2\, dW$$ **, λ=3; 3.5 and 4**









q-Gaussian distribution

$$\mathrm{d}x = \left(\eta - \frac{1}{2}\lambda\right)\left(x_m^2 + x^2\right)^{\eta-1} x\,\mathrm{d}t + \left(x_m^2 + x^2\right)^{\eta/2}\mathrm{d}W, \quad \eta > 1, \quad \lambda > 1$$

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\lambda-1}{2}\right) x_m}\left(\frac{x_m^2}{x_m^2 + x^2}\right)^{\lambda/2} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\lambda-1}{2}\right) x_m}\exp_q\left\{-\lambda\frac{x^2}{2x_m^2}\right\}$$

Regular distribution of signal for $x > 0$, $x = 0$ and $x < 0$.

$$S(f) = \frac{A}{(f_0^2 + f^2)^{\beta/2}} = \exp_q\left\{-\beta\frac{f^2}{2f_0^2}\right\} \qquad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}$$

$$C(s) = \int_0^\infty S(f)\cos(2\pi f s)\,\mathrm{d}f = \frac{A\sqrt{\pi}}{\Gamma(\beta/2)}\left(\frac{\pi s}{f_0}\right)^h K_h(2\pi f_0 s)$$

$$F(s) = F_2^2(s) = \left\langle |x(t+s) - x(t)|^2\right\rangle = 2[C(0) - C(s)] = 4\int_0^\infty S(f)\sin^2(\pi s f)\,\mathrm{d}f.$$

- ✓ **J. Ruseckas and B.K., Phys. Rev. E 84, 051125 (2011)**

Herding model and 1/f noise

- The nonlinear **SDE**s provide macroscopic description of a complex system.
- The microscopic, agent based reasoning of equations exhibiting
- 1/f noise can yield further insights into behavior of the system.
- Here we show that it is possible to obtain the nonlinear **SDE** of the similar form starting from agent-based herding model.

Abstract – We provide evidence that for some values of the parameters a simple agent-based model, describing herding behavior, yields signals with $1/f$ power spectral density. We derive a non-linear stochastic differential equation for the ratio of number of agents and show, that it has the form proposed earlier for modeling of $1/f^{\beta}$ noise with different exponents β. The non-linear terms in the transition probabilities, quantifying the herding behavior, are crucial to the appearance of $1/f$ noise. Thus, the herding dynamics can be seen as a microscopic explanation of the proposed non-linear stochastic differential equations generating signals with $1/f^{\beta}$ spectrum.

J. Ruseckas[a], B. Kaulakys and V. Gontis, EPL, **96** (2011) 60007

Kirman's model

We start from the Kirman's seminal herding agent-based model:

- **A. Kirman, Epidemics of opinion and speculative bubbles in financial markets, In Money and Financial Markets,1991, p. 354.**
- **A. Kirman, Q. J. Econ., 108 (1993) 137.**

- **It is worth to notice that the appropriate agent-based models can yield emergence**
- **the power-law scaling,**
- **long-range correlations,**
- **(multi)fractality**
- **and fat tails,**

However the omnipresent 1/f noise have not yet been revealed in such approach.

- **In the model the dynamic evolution is described as a Markov chain.**
- **There is a fixed number N of agents,**
- **Each of them being in state 1 or in state 2.**
- **The number of agents in the first state is denoted by *n*,**
- **and the number in the second state by *N-n*.**
- **Describing the dynamics as a jump Markov process in continuous time,**
- **The transition probabilities per unit time are given by Eqs. for one-step stochastic process**

$$p(n \to n+1) \equiv p^{+}(n) = (N-n)(\sigma_1 + hn), \qquad (1)$$

$$p(n \to n-1) \equiv p^{-}(n) = n(\sigma_2 + h(N-n)). \qquad (2)$$

Constants σ_1 and σ_2 describe the typical tendency to change the state, while the term *h* describes the herding tendency.

The transition probabilities imply the **Master equation**

$$\frac{\partial}{\partial t}P_x(x,t) = -\frac{\partial}{\partial x}h(\varepsilon_1(1-x)-\varepsilon_2 x)P_x(x,t)$$

$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}h\left(2x(1-x)+\frac{\varepsilon_1}{N}(1-x)+\frac{\varepsilon_2}{N}x\right)P_x(x,t), \qquad (4)$$

for the probability $P_n(t)$ to find n agents in the state 1 at time t.

For large enough N we can represent the dynamics by a continuous variable $x=n/N$.

A Fokker-Planck equation, derived from the Master equation (4), assuming that N is large and neglecting the terms of the order of $1/N^2$ is

$$\frac{\partial}{\partial t}P_n = p^+(n-1)P_{n-1} + p^-(n+1)P_{n+1}$$

$$-(p^+(n)+p^-(n))P_n.$$

where $\varepsilon_1 \equiv \sigma_1/h$, $\varepsilon_2 \equiv \sigma_2/h$ are scaled parameters.

This Fokker-Planck equation corresponds to the stochastic differential equation

$$dx = h(\varepsilon_1(1-x) - \varepsilon_2 x)dt + \sqrt{2hx(1-x)}dW, \qquad (6)$$

Introduction of the new variable *y*, i.e.,

the ratio of the number of agents in the state 2 to the number of agents in the state 1

$$y = \frac{1-x}{x} = \frac{N-n}{n}$$

yields the nonlinear SDE for the ratio of number of agents in two states

$$dy = h[(2-\varepsilon_1)y + \varepsilon_2](1+y)dt + \sqrt{2hy}(1+y)dW.$$

For *y* >> 1 we have the approximate form

$$dy \approx h(2-\varepsilon_1)y^2 dt + \sqrt{2h}y^{\frac{3}{2}}dW.$$

This equation

$$\mathrm{d}y \approx h(2-\varepsilon_1)y^2\mathrm{d}t + \sqrt{2h}y^{\frac{3}{2}}\mathrm{d}W.$$

is a special case of our general equation for 1/f noise

$$\mathrm{d}x = \sigma^2\left(\eta - \frac{1}{2}\lambda\right)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W.$$

generating signals with the power spectral density

$$S(f) \sim \frac{1}{f^{\beta}}, \qquad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}.$$

Special cases of the equation:

i) η =0 and σ =1,

$$\mathrm{d}x = \frac{\delta - 1}{2}\frac{1}{x}\mathrm{d}t + \mathrm{d}W,$$

***Bessel process* of dimension δ =1−λ.**

ii) η =1/2 and σ =2,

$$\mathrm{d}x = \delta\mathrm{d}t + 2\sqrt{x}\mathrm{d}W,$$

***Squared Bessel process* of dimension δ = 2(1−λ).**

iii) with exponential restriction for η =1/2, x_{min} =0 and m = 1,

$$\mathrm{d}x = k(\theta - x)\mathrm{d}t + \sigma\sqrt{x}\,\mathrm{d}W,$$

***Cox-Ingersoll-Ross* (CIR) process.**

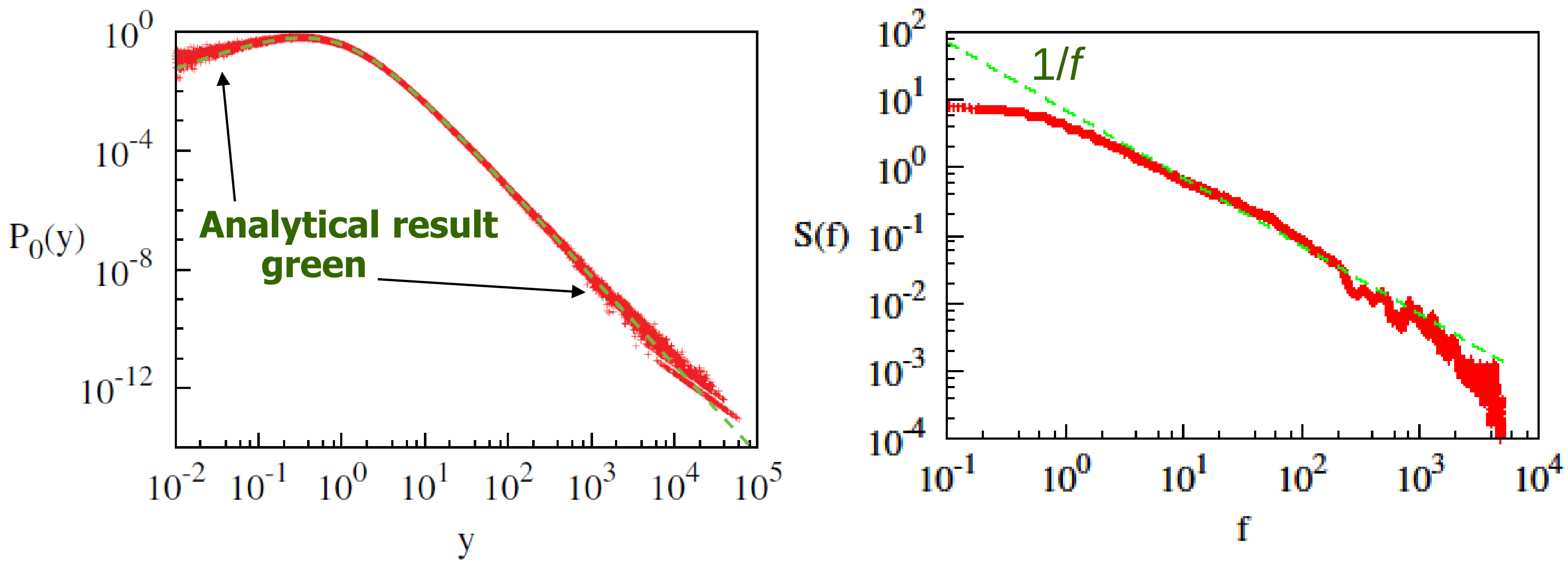
iv) with exponential restriction for x_{max} =∞ and m=2η −2,

$$\mathrm{d}x = \mu x\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W,$$

***Constant Elasticity of Variance* (CEV) process.**

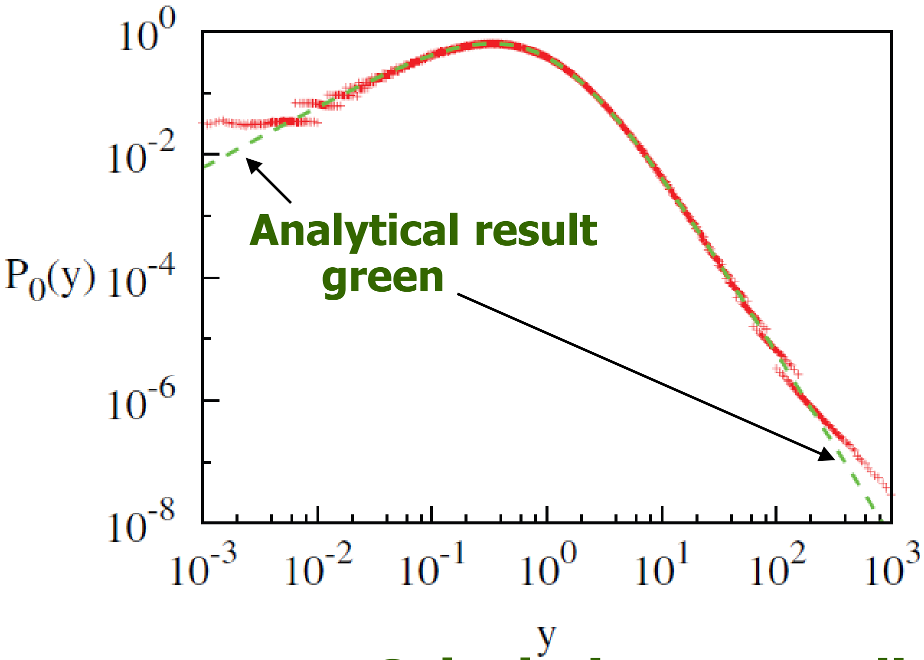
Numerical analysis of equation

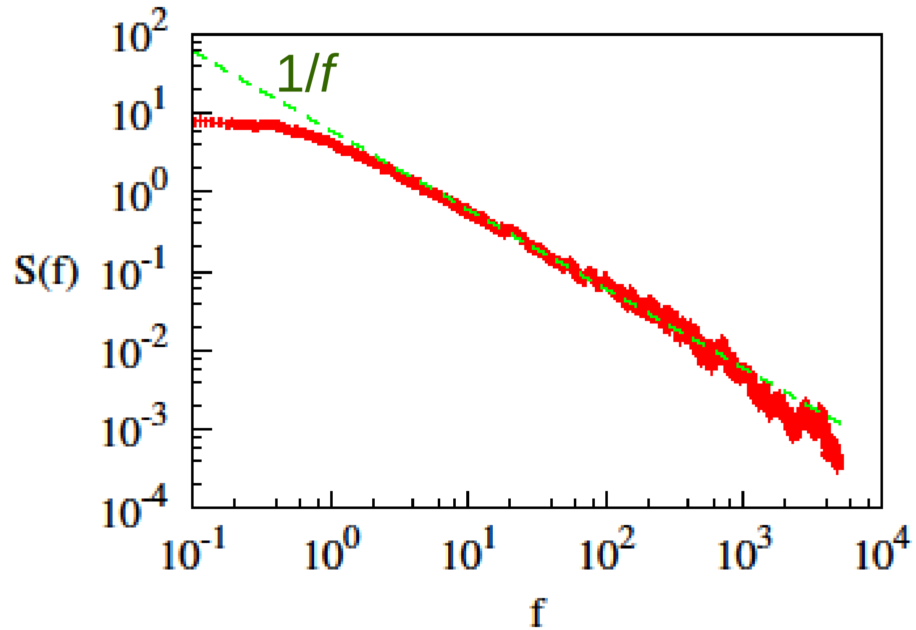
$$\mathrm{d}y = h[(2-\varepsilon_1)y+\varepsilon_2](1+y)\mathrm{d}t+\sqrt{2hy}(1+y)\mathrm{d}W.$$



Comparison with analytical expressions of the numerical steady-state PDF, $P_0(y)$, and the power spectral density, $S(f)$, of the signal generated by this equation

Comparison with the microscopic agent model





Calculations according to equations:

$$p(n \to n+1) \equiv p^{+}(n) = (N-n)(\sigma_1 + hn), \qquad (1)$$

$$p(n \to n-1) \equiv p^{-}(n) = n(\sigma_2 + h(N-n)). \qquad (2)$$

for $y = \frac{1-x}{x} = \frac{N-n}{n}.$

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Possible generalizations (1)

For the stochastic variable $y = \left(\frac{1-x}{x}\right)^{1/\alpha}$ **SDE is**

$$\mathrm{d}y = \frac{h}{\alpha}\left[\left(1+\tfrac{1}{\alpha}-\varepsilon_1\right)+\left(\varepsilon_2+\tfrac{1}{\alpha}-1\right)y^{-\alpha}\right]y(1+y^{\alpha})\mathrm{d}t + \frac{\sqrt{2h}}{\alpha}y^{1-\frac{\alpha}{2}}(1+y^{\alpha})\mathrm{d}W. \quad (23)$$

The corresponding steady-state PDF is

$$P_0(y) = \frac{\alpha\Gamma(\varepsilon_1+\varepsilon_2)}{\Gamma(\varepsilon_2)\Gamma(\varepsilon_1)}\frac{y^{\alpha\varepsilon_2-1}}{(1+y^{\alpha})^{\varepsilon_2+\varepsilon_1}}. \quad (24)$$

For the parameters α=1 and ε_2 =1 Eq. (24) corresponds to *q*-exponential distribution with *q* =1+1/(1+ε_1),

while for the parameters α=2 and ε_2 =1/2 it corresponds to *q*-Gaussian distribution with *q* =1+2/(1+2ε_1).

Possible generalizations (2)

Rate at which the agents meet depends on the global state of the system.

The new transition probabilities are:

$$p(n \to n+1) = \frac{1}{\tau(n)}(N-n)(\sigma_1 + hn),$$

$$p(n \to n-1) = \frac{1}{\tau(n)} n(\sigma_2 + h(N-n)),$$

$\tau(n)$ describes the time scale of the microscopic events.

For $\tau(y) = y^{-\gamma}$

SDE for the variable $y = (1-x)/x$

$$\mathrm{d}y = h[(2-\varepsilon_1)y + \varepsilon_2]y^{\gamma}(1+y)\mathrm{d}t + \sqrt{2hy^{1+\gamma}}(1+y)\mathrm{d}W$$

generates signals with power spectral density

$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)} = 1 + \frac{\varepsilon_1 + \gamma - 2}{1 + \gamma}.$$

Some conclusions

- **Nonlinear stochastic differential equation generating $1/f^{\beta}$ noise may be obtained from the microscopic agent-based herding model.**
- **The nonlinear terms in the transition probabilities, quantifying the herding behavior, are crucial to the appearance of 1/f noise.**
- **The herding dynamics can be seen as a microscopic explanation of the proposed non-linear stochastic differential equations generating signals with $1/f^{\beta}$ spectrum.**