Bursting behavior of non-linear stochastic model and empirical high-frequency return

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Abstract

Recently we have proposed a non-linear stochastic model reproducing power law probability and spectral densities [1, 2]. The reproduced statistical properties match ones observed in the timeseries of high frequency absolute returns of financial markets. The proposed model and its generalizations [3] also exhibit power law bursting behavior [4, 5]. Mathematically bursting behavior might be backed by known results in hitting time statistics of known simple models (ex. Brownian motion, geometric Brownian motion and etc.). In this contribution we aim to show that bursting behavior is also observed in the high-frequency empirical financial market data, thus the proposed model might be of particular interest due to possible applications of burst statistics towards risk management.

Burst statistics of the time series

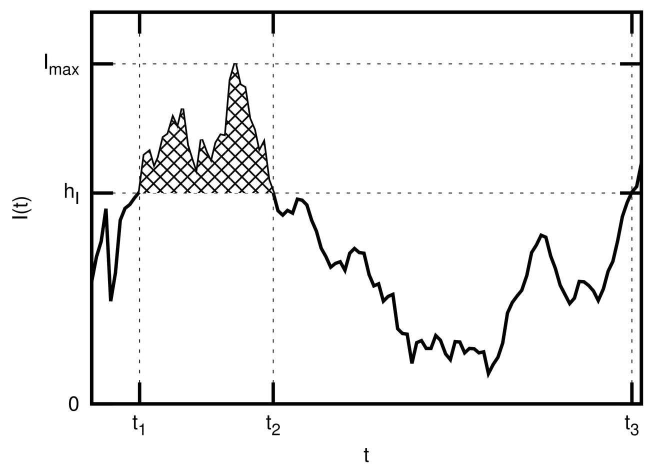

Figure 1: Time series exhibiting bursty behavior, $I(t)$. Here h_I is threshold value, above which bursts are detected. t_i are the three visible threshold passage events. I_{max} is the highlighted burst's peak value. The highlighted area is called burst size, S. The other relevant statistical properties are defined as: $T = t_2 - t_1$ (burst duration), $\theta = t_3 - t_2$ (inter-burst time) and $\tau = T + \theta = t_3 - t_1$ (waiting time).

The considered models

Previously [1, 2] we have proposed a double stochastic model model driven by the non-linear SDE,

$$\mathrm{d}x = \left[\eta - \frac{\lambda}{2} - \left(\frac{x}{x_{max}}\right)^2\right] \frac{(1+x^2)^{\eta-1}}{(\epsilon\sqrt{1+x^2}+1)^2} x \mathrm{d}t_s + \frac{(1+x^2)^{\frac{\eta}{2}}}{\epsilon\sqrt{1+x^2}+1} \mathrm{d}W_s, \qquad (1)$$

and modulated by the q-Gaussian noise with $\lambda = 5$ and $r_0 = 1 + \frac{\bar{r}_0}{\tau_s}\left|\int_{t_s}^{t_s+\tau_s} x(s)\mathrm{d}s\right|$. This double stochastic model enables the reproduction of two main stylized facts related to the absolute return - power law probability density function and fractured spectral density.

The proposed double stochastic model appears to be too complex to study analytically. Yet to understand the bursty behavior of the model and financial markets themselves we can study are more simple stochastic model driven by a simpler SDE

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1}\mathrm{d}t_s + x^{\eta}\mathrm{d}W_s. \qquad (2)$$

This SDE posses very similar statistical features - it is able to reproduce power law probability density, but its spectral density consists only of one power law: $S(f) \sim \frac{1}{f^\beta}$, $\beta = 1 + \frac{\lambda - 3}{2(\eta-1)}$.

The understanding of burst dynamics of the (2) also proves useful as in certain cases it can be reduced to widely known and used stochastic processes, namely Bessel process ($\eta = 0$, $\sigma = 1$), squared Bessel process ($\eta = 1/2$, $\sigma = 2$), CIR (add linear restriction from the top, $\eta = 1/2$) and CEV processes (add restriction from bottom with $m = 2\eta - 2$, $\lambda = 2\eta$).

Obtaining burst duration

In order to obtain the analytic expression of the burst duration PDF we assume that the burst duration is the same as the first hitting time of the stochastic process starting infinitesimally near the threshold. As we are willing to use known results of the first hitting times, we have to transform (2) into the other known stochastic process. The Bessel process appears to be the best choice:

$$x^\eta \partial_x y(x) = \pm 1, \quad \Rightarrow \quad y(x) = \frac{1}{(\eta-1)x^{\eta-1}}, \quad \Rightarrow \quad \mathrm{d}y = \left(\nu + \frac{1}{2}\right)\frac{\mathrm{d}t_s}{y} + \mathrm{d}W_s, \; \nu = \frac{\lambda - 2\eta + 1}{2\eta - 2}. \qquad (3)$$

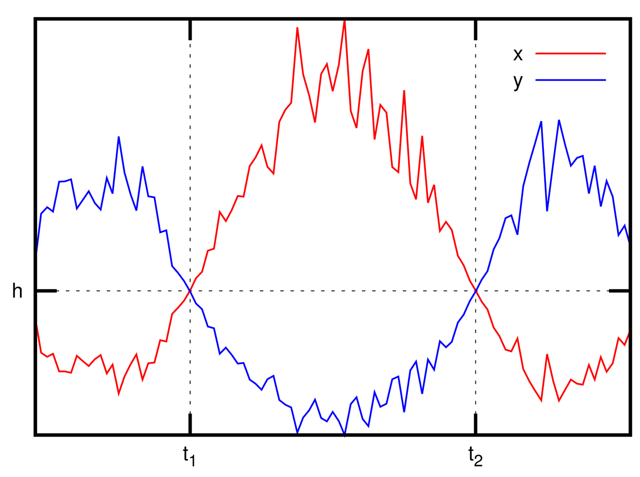

Figure 2: The burst duration in x time series is the inter-burst time in y time series.

From the transformation, see Fig. 2, follows that we have to use the result for inter-burst time of the Bessel process, which is given in [6]:

$$\rho^{(\nu)}_{y_0,h_y} = \frac{h_y^{\nu-2}}{y_0^{\nu}} \sum_{k=1}^{\infty} \frac{j_{\nu,k} J_\nu\left(\frac{y_0}{h_y} j_{\nu,k}\right)}{J_{\nu+1}(j_{\nu,k})} \exp\left(-\frac{j_{\nu,k}^2}{2} t\right), \quad p^{(\nu)}_{h_y}(t) = \lim_{y_0 \to h_y} \frac{\rho^{(\nu)}_{y_0,h_y}(t)}{h_y - y_0}. \qquad (4)$$

To evaluate the limit we have to expand $J_\nu\left(\frac{y_0}{h_y} j_{\nu,k}\right)$ near $\frac{y_0}{h_y} = 1$:

$$J_\nu\left(\frac{y_0}{h_y} j_{\nu,k}\right) \approx \left(1 - \frac{y_0}{h_y}\right) j_{\nu,k} J_{\nu+1}(j_{\nu,k}), \quad \Rightarrow \quad p^{(\nu)}_{h_y}(t) \approx C_1 \sum_{k=1}^{\infty} j_{\nu,k}^2 \exp\left(-\frac{j_{\nu,k}^2 t}{2h_y^2}\right). \qquad (5)$$

Since $j_{\nu,k}$ are almost equally spaced, we can replace the sum by integration, which yields:

$$p^{(\nu)}_{h_y}(t) \approx C_2 \left[\frac{h_y^2 j_{\nu,1}}{t} \exp\left(-\frac{j_{\nu,1}^2 t}{2h_y^2}\right) + \sqrt{\frac{\pi}{2}} \frac{h_y^3}{t^{3/2}} \mathrm{erfc}\left(\frac{j_{\nu,1}\sqrt{t}}{\sqrt{2} h_y}\right)\right]. \qquad (6)$$

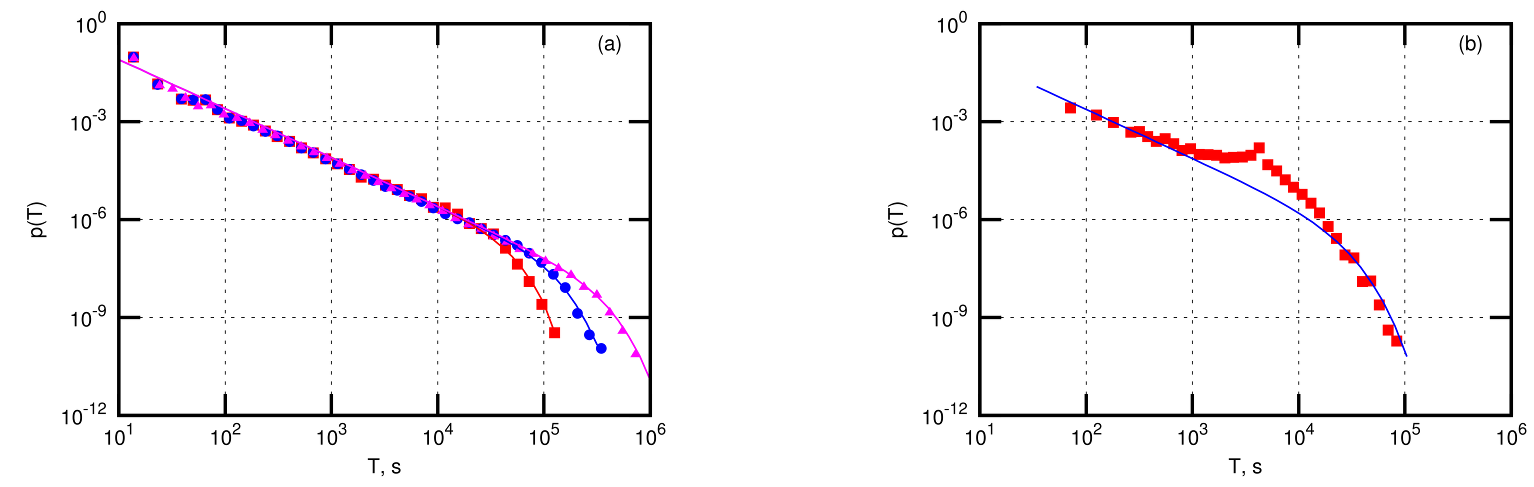

Figure 3: Fitting model (a) and empirical (b) data with (6). Model parameters were set as follows: $\lambda = 4$ (all three cases), $\eta = 2.5$ (red squares), 2 (blue circles) and 1.5 (magenta triangles). Model data fitted using $\nu = 0$ (red curve), 0.5 (blue curve), 2 (magenta curve). Empirical data fitted assuming $\nu = -0.2$.

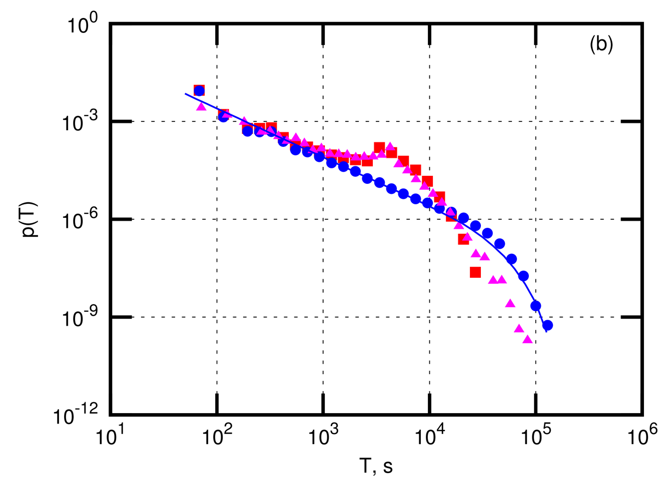

Figure 4: Comparison of the empirical data (red squares), data obtained by solving a more complex SDE (blue circles) and data obtained from the double stochastic model (magenta triangles). Model parameters were set as follows: $\eta = 2.5$, $\lambda = 3.6$, $x_{max} = 10^3$, $\epsilon = 0.017$, $\bar{r}_0 = 0.4$. Data obtained by solving a more complex SDE fitted assuming that $\nu = 0$ (blue curve).

Statistical properties of the other burst related variables

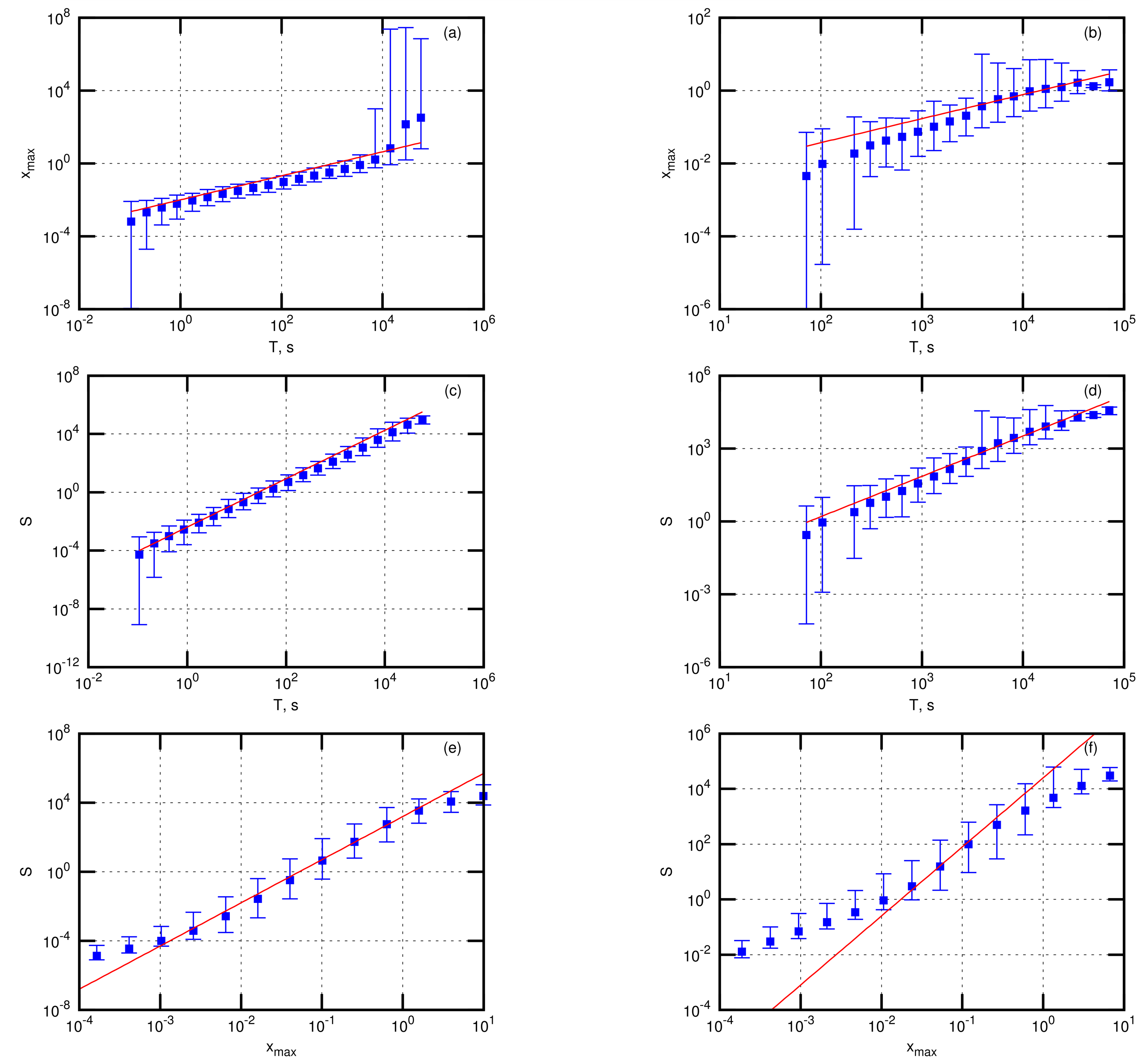

Figure 5: Scatter plots of three burst related variables, T, S and x_{max}, observed in the simple model ((a), (c), (e)) and empirical data ((b), (d), (f)). Curves provide power law fits with the exponents: $\alpha = 0.66$ ((a), (b)), 1.66 ((c), (d)), 2.5 ((e), (f))

Conclusions

- Simple stochastic model reproduces empirical burst statistics rather well (see Fig. 5).
- Double stochastic model may be used to precisely reproduce burst duration PDF (see Fig. 3 and 4). Note that the secondary noise process is of the utmost importance.
- Burst statistics of absolute return can be modeled using non-linear SDE with $\eta > 1$.

Acknowledgments

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