Agent-based modeling

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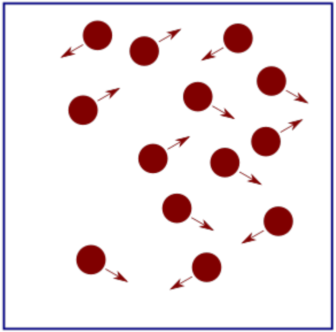
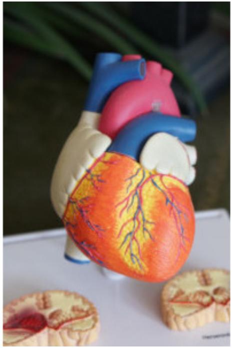
Complex Physical and Social Systems Group

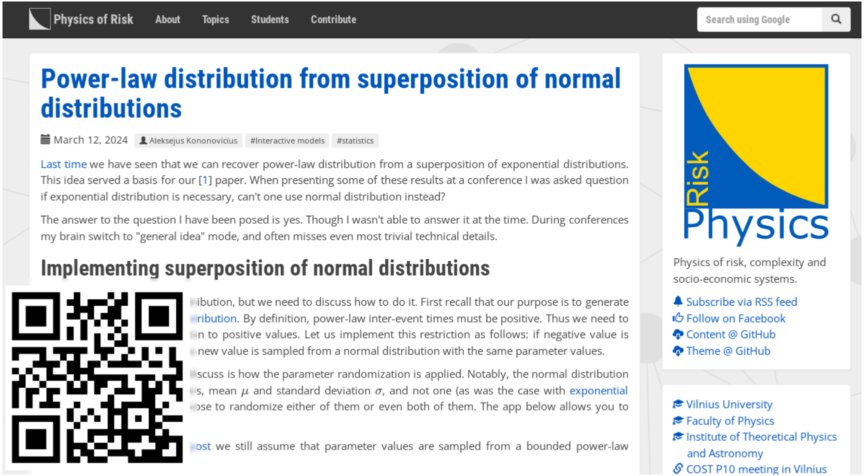




Areas of interest: nonlinear dynamics and synchronization, long-range memory, physics of socio-economic systems.

Images: unsplash.com, KAL's cartoons

Physics of Risk blog



Physics of Risk About Topics Students Contribute

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Power-law distribution from superposition of normal distributions

March 12, 2024 Aleksejus Kononovicius #interactive models #statistics

Last time we have seen that we can recover power-law distribution from a superposition of exponential distributions. This idea served a basis for our [1] paper. When presenting some of these results at a conference I was asked question if exponential distribution is necessary, can't one use normal distribution instead?

The answer to the question I have been posed is yes. Though I wasn't able to answer it at the time. During conferences my brain switch to "general idea" mode, and often misses even most trivial technical details.

Implementing superposition of normal distributions

ibution, but we need to discuss how to do it. First recall that our purpose is to generate ribution. By definition, power-law inter-event times must be positive. Thus we need to n to positive values. Let us implement this restriction as follows: if negative value is new value is sampled from a normal distribution with the same parameter values.

scuss is how the parameter randomization is applied. Notably, the normal distribution s, mean μ and standard deviation σ, and not one (as was the case with exponential ose to randomize either of them or even both of them. The app below allows you to

ost we still assume that parameter values are sampled from a bounded power-law

Physics of risk, complexity and socio-economic systems.

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The plan

1. General premise
2. Rational agents and game theory
3. Wealth and ideal gasses
4. Network science
5. Opinion dynamics

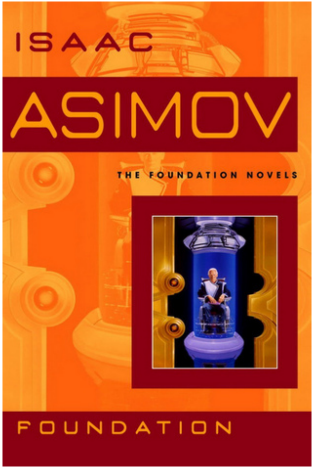


Image: goodreads.com

General premise

Agent. . . what?

Models generalize reality.

Agents:

- represent us, or groups of us,
- have defining features, and behaviors,
- may have goals,
- observe and interact with environment,
- observe and interact with peers.

"All models are wrong, but some are useful" (George Box)

Image: MoviePosterDB.com

Do we really need another modeling framework?

- **Agents:** passengers
- **Environment:** plane (aisle, seats, storage)

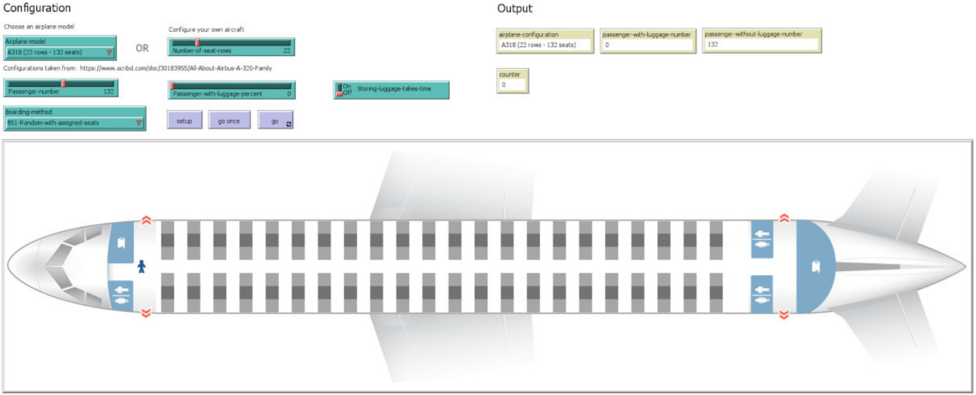
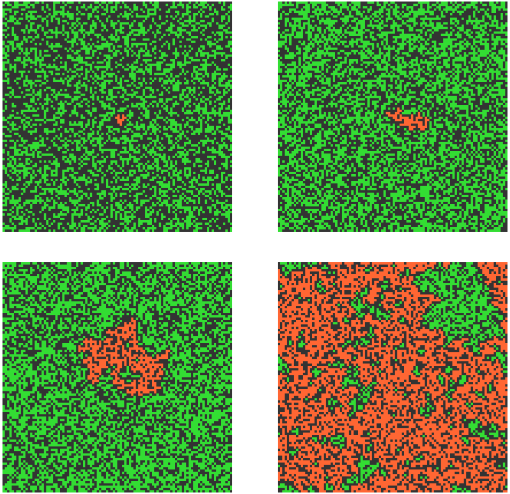


Figure: [Delcia *et al.*(2018)]

Physics? Really?



Forest fire model:

- Forest: ρ density of trees
- Fire: spreads to neighboring trees
- How big will the fire be?

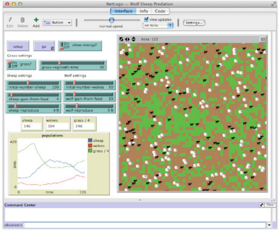
(⇐): $\rho_{\{\nwarrow,\nearrow,\swarrow,\searrow\}} = \{0.4, 0.5, 0.55, 0.6\}$.

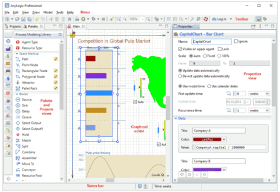
Interactive app: Physics of Risk

Specialized tools

- NetLogo - approachable, custom language
- GAMA - GIS, custom language + Java
- AnyLogic - used in the industry

- Agents.jl - Julia
- Mesa - Python
- Mason - Java, supports GIS
- Repast - Java, supports HPC





Review: [Antelmi *et al.*(2022)]; Images: Wikimedia, Anylogic

Rational agents and game theory

Game theory

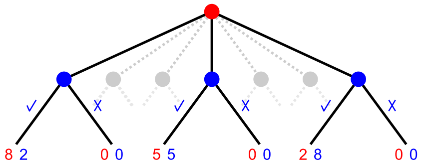
Explores interactions between rational and self-interested agents.

Games:

- cooperative or competitive
- (non-)zero sum
- (a)symmetric
- (a)synchronous
- (in)finite
- …

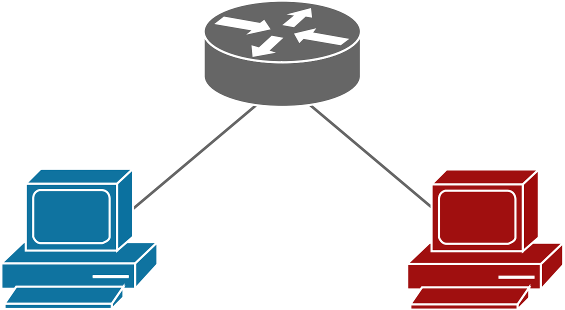
	R	P	S
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Payoff matrix for a r-p-s game



Decision tree of an ultimatum game

Pure strategies (in the TCP backoff game)



	B	C
Back-off	-1, -1	-4, 0
Continue	0, -4	-3, -3

- What is optimal?
- What is rational?

Icons: vecta.io

Some games have no pure strategy...

GK \ Taker	L	R
Left	1, 0	0, 1
Right	0, 1	1, 0

Matching pennies game

But there might be a mixed strategy. To find it you need to make your opponent not care about their action.

Image: Wikimedia

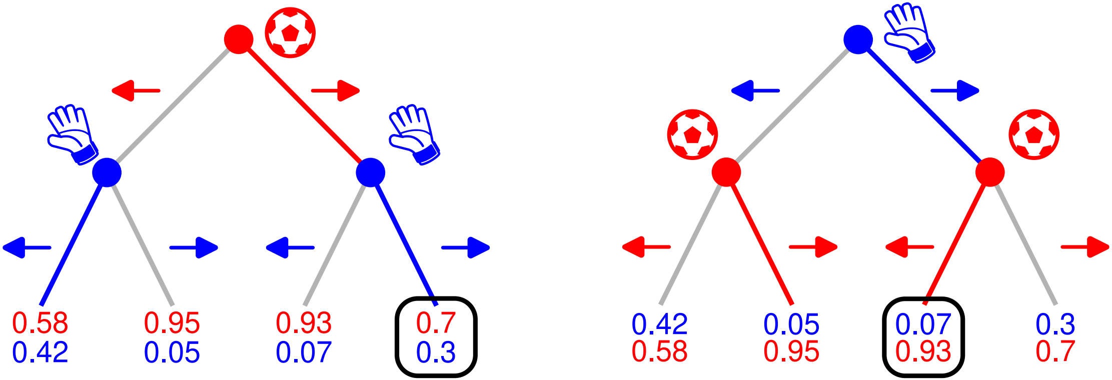
A practical problem for a football manager...

GK \ Taker	L	R
Left	0.42, 0.58	0.07, 0.93
Right	0.05, 0.95	0.3, 0.7

1. Should GK jump left? (Answer: $p_{GK} \approx 0.42$)
2. Should penalty taker shoot left? (Answer: $p_T \approx 0.38$)
3. Expected outcome? (Answer: $U_{GK} \approx 0.2$)

Article: [Palacios-Huerta (2003)]; Image: sportingnews.com

Solution: moneyball reaction



Quick GK (left) or quick taker (right).

Icons: "Eleven: Football Manager Board Game" (by Portal Games)

Delving deeper

- More actions
- More players
- Consecutive games
- Random games
- Behavioral rationality

Designing:

- Auctions
- Voting
- ...

Resilience:

- Errors
- Manipulations

Recommendation: “Game Theory” course (Coursera and Youtube)

Wealth and ideal gasses

Kinetic exchange model

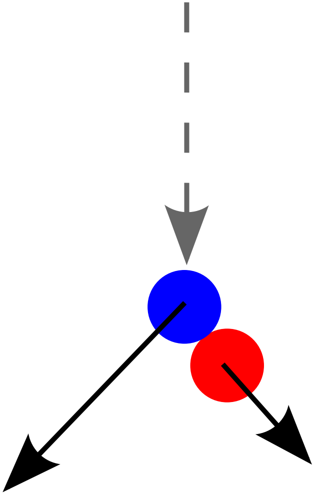
1. Two particles i and j collide.
2. Δw_{ij} energy is transferred:

$$\Delta w_{ij} = r_i w_i - r_j w_j .$$

3. Updating energies:

$$w_i\left(t+1\right) = w_i\left(t\right) - \Delta w_{ij},$$

$$w_j\left(t+1\right) = w_j\left(t\right) + \Delta w_{ij}.$$



Empirical wealth data

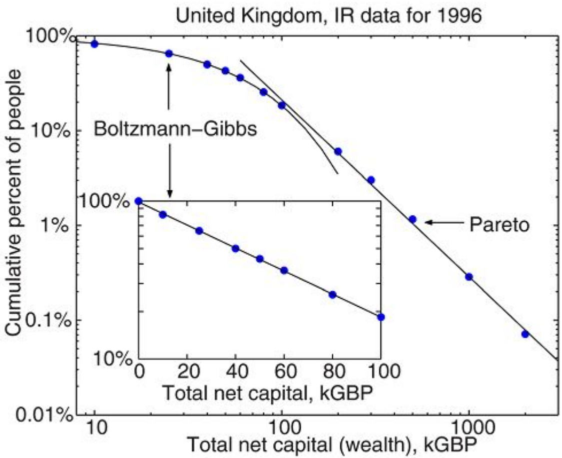


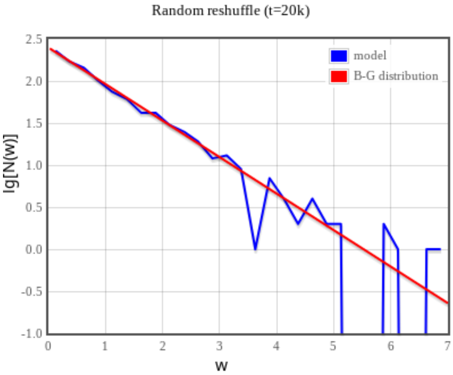
Figure: [Dragulescu and Yakovenko (2001)]

Simplest kinetic exchange model

1. Two random agents i and j meet.
2. Wealth is split randomly,

$$\Delta w_{ij} = (1 - \varepsilon)\, w_i - \varepsilon w_j,$$

   with $\varepsilon \sim \mathcal{U}(0, 1)$.
3. Update wealth.



Interactive app: Physics of Risk

Analytical approach to the model

- The master equation:

$$\frac{\partial p(w,t)}{\partial t} = \frac{\partial N^+(w,t)}{\partial t} - \frac{\partial N^-(w,t)}{\partial t}$$

- Counting "leaving" agents: $\frac{\partial N^-(w,t)}{\partial t} \sim 2p(w,t)$
- Counting "arriving" agents: $\frac{\partial N^+(w,t)}{\partial t} \sim 2\mathbb{P}\left[0 < w < w_i(t) + w_j(t)\right]$
- We care about stationary distribution:

$$\frac{\partial p_{st}(w)}{\partial t} = 0 \quad \Rightarrow \quad p_{st} = \mathbb{P}_{st}\left[\ldots\right] \quad \Rightarrow \quad p_{st}(w) = \frac{1}{\langle w \rangle} \exp\left(-\frac{w}{\langle w \rangle}\right).$$

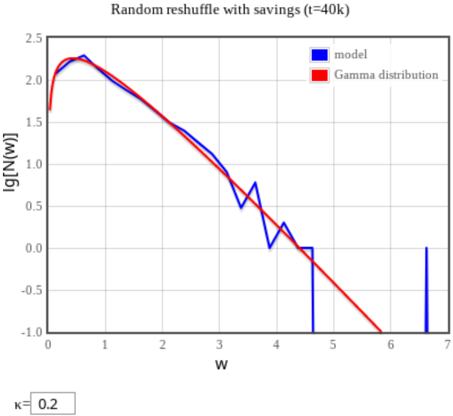
Article: [Calbet *et al.*(2011)]

Introducing saving propensity

1. Two random agents i and j meet.
2. They reserve κ fraction of their wealth. Remaining wealth is split randomly,

$$\Delta w_{ij} = (1-\kappa)\left[(1-\varepsilon)\, w_i - \varepsilon w_j\right].$$

with $\varepsilon \sim \mathcal{U}(0,1)$.
3. Update wealth.



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Deriving moments

By definition, lhs and rhs should be equal in distribution:

$$w_i(t+1) \overset{d}{=} \kappa w_i(t) + \varepsilon(1-\kappa)\left[w_i(t) + w_j(t)\right]$$

Thus, for the m-th raw moment of a stationary distribution:

$$\langle w^m \rangle = \langle \{\kappa w_i + \varepsilon(1-\kappa)[w_i + w_j]\}^m \rangle .$$

Needs to be solved recurrently:

$$\left\langle w^1 \right\rangle = 1,$$

$$\left\langle w^2 \right\rangle = \frac{\kappa+2}{1+2\kappa},$$

$$\left\langle w^3 \right\rangle = \frac{3(\kappa+2)}{(1+2\kappa)^2},$$

$$\left\langle w^4 \right\rangle = \frac{72+12\kappa-2\kappa^2+9\kappa^3-\kappa^5}{(1+2\kappa)^2(3+6\kappa-\kappa^2+2\kappa^3)}.$$

Suggest decent approximation

$$p(w) \sim w^{n-1} \exp(-nw),$$

with $n = 1 + \frac{3\kappa}{1-\kappa}$.

Constructing power-law distribution

It is easy to show that $(0 < \alpha < 2)$:

$$\int_0^\infty \left[\frac{1}{\lambda^\alpha} \cdot \lambda \exp\left(-\lambda x\right)\right] d\lambda = \frac{1}{x^{2-\alpha}}.$$

But for wealth distribution,

$$\int_0^1 \left\{p(\kappa) \cdot w^{n(\kappa)-1} \exp\left[-n(\kappa) w\right]\right\} d\kappa \propto \frac{1}{w^2},$$

$$p(\kappa) = \,???$$

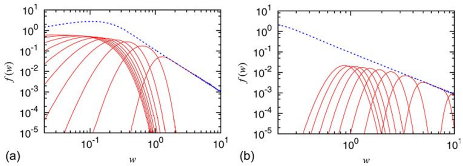


Fig. 3. (Color online) Wealth distribution $f(w)$ for uniformly distributed κ_i (or λ_i) in the interval (0,1); $f(w)$ is decomposed into partial distributions $f_i(w)$, where each $f_i(w)$ is obtained by counting the statistics of those agents with parameter λ_i in a specific sub-interval (from Ref. 36). The left panel shows the decomposition of $f(w)$ into ten partial distributions in the λ-subintervals (0, 0.1), (0.1, 0.2),..., (0.9, 1). The right panel decomposes the final partial distribution in the λ-interval (0.9, 1) into partial distributions obtained by counting the statistics of agents with λ-subintervals (0.9, 0.91), (0.91, 0.92),..., (0.99, 1). Note how the power law appears as a consequence of the superposition of the partial distributions.

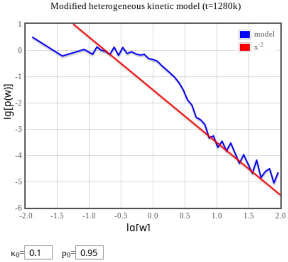
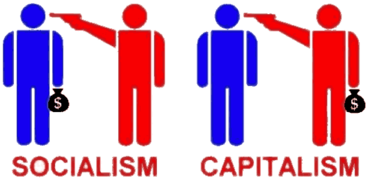


Figure: [Patriarca and Chakraborti (2013)]; Interactive app: Physics of Risk

Delving deeper

Wealth / income:

- Compatibility with Economics
- Skills and luck
- Temporal dynamics
- Realistic income mechanisms



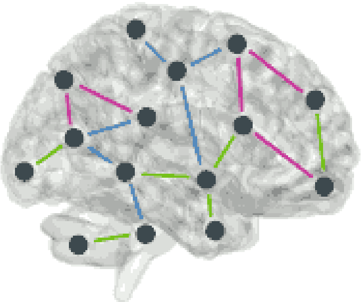
But not only wealth / income:

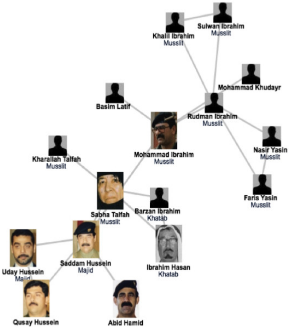
- Opinion dynamics (Biswas-Chatterjee-Sen model)
- Designing ranking systems (ELO)
- Epidemiological models
- Alcohol consumption

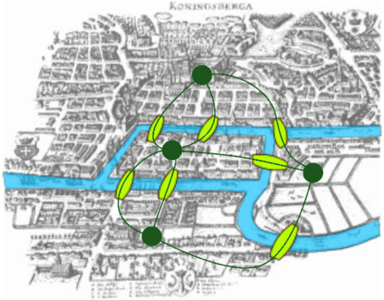
Reviews: [Patriarca and Chakraborti (2013)], [Toscani *et al.*(2022)]; Image: meme (politifake.org)

Network science

Connections



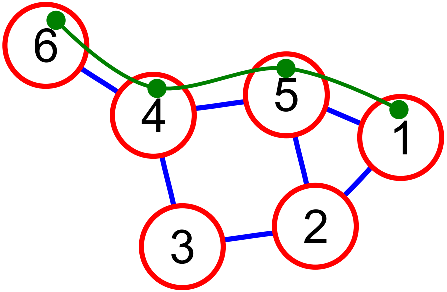




Images: [Lynn and Basset (2019)], slate.com, Wikimedia.

Main terminology

Network is a collection of **nodes** and **links**. Mathematicians prefer terms **graph**, **vertex** and **edge**.



- **Neighboring** nodes - connected by edges.
- Node's **degree** - a number of its neighbors.
- **Path** - sequence of neighboring nodes.
- **Geodesic** - shortest $i \to j$ path.
- **Diameter** - longest geodesic in a network.
- …

Adjacency matrix

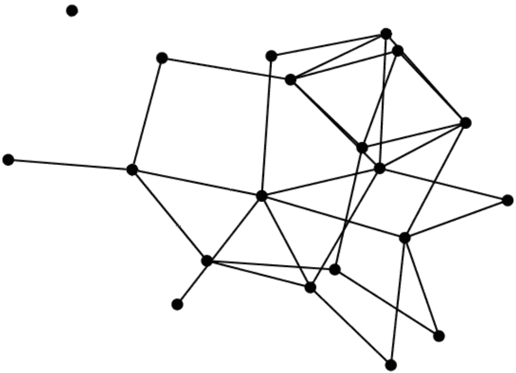
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- If $A_{ij} \neq 0$, then there exists an edge pointing from j to i.
- Node degree:
  $k_i = \sum_{j=1}^{N} \mathbf{1}_{A_{ij} \neq 0} = \sum_{j=1}^{N} \mathbf{1}_{A_{ji} \neq 0}$.
- $(\mathbf{A}^m)_{ij}$ counts all $j \to i$ paths.

Specific links can be

- **looping**, if $A_{ii} = 1$.
- **directional**, if $A_{ij} \neq A_{ji}$.
- **multiple**, if $A_{ij} \in \mathbb{N}_0$.
- **weighted**, if $A_{ij} \in \mathbb{R}$.

Erdos-Renyi (random) network



1. Start with N nodes and $L = 0$ edges.
2. Iterate over all possible pairs. Add edge with probability p.

Edges on average:

$$\langle L \rangle = pN\left(N-1\right)/2.$$

Average degree:

$$\langle k \rangle = 2L/N = p\left(N-1\right).$$

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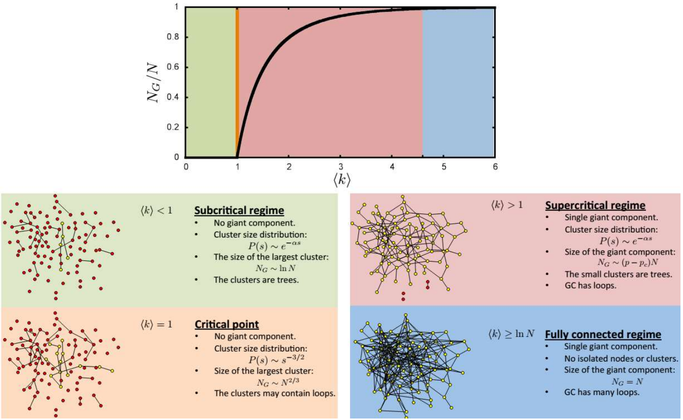
Phase transition in E-R network



If node i belongs to **giant component**, then its neighbor j is also in it.

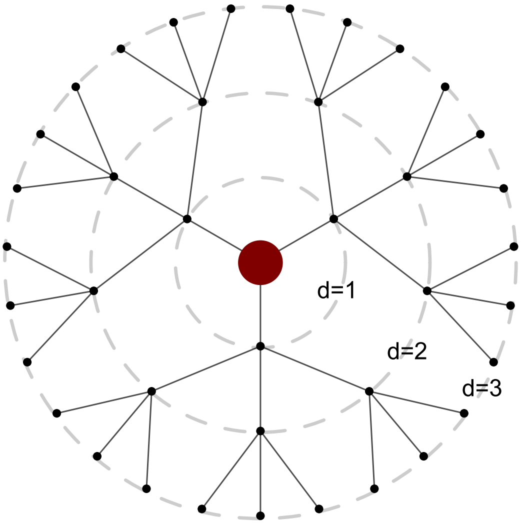
Probability to not be in g.c.:

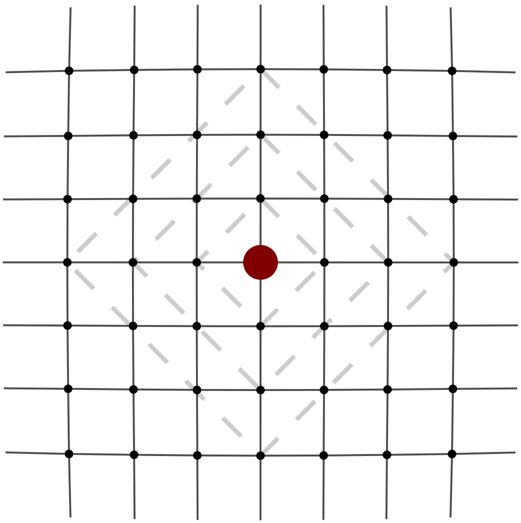
$$u = [1 - (1 - u)\, p]^{N-1},$$

$$\frac{N_G}{N} = 1 - \exp\left[- \langle k \rangle \frac{N_G}{N} \right].$$

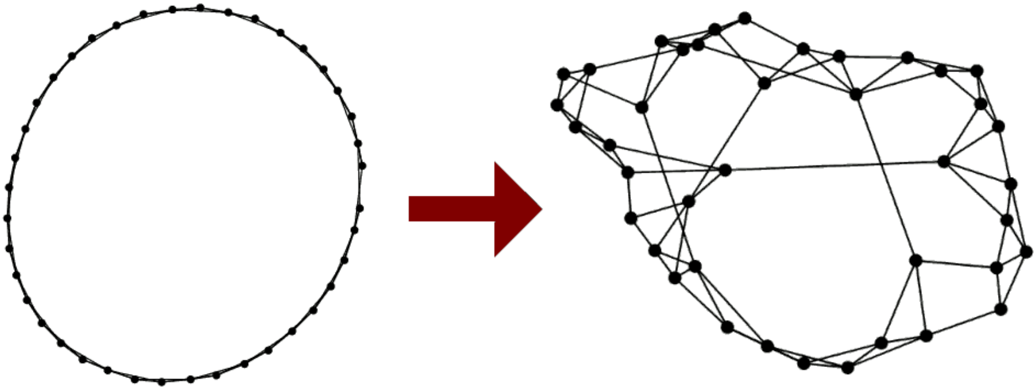
Figures: networksciencebook.com (V26 edition)

Randomness creates reach





Watts-Strogatz network



W-S network: Introduce random edges into a regular structure.

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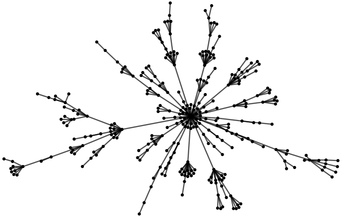
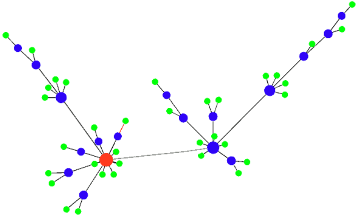
Scale-free networks

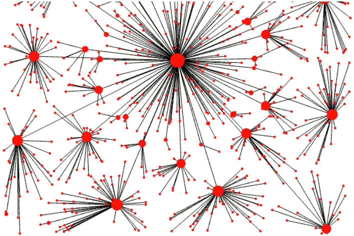


Preferential attachment:

$$p(i \to j) = \frac{k_j}{\sum_m k_m}$$



Edge redirection: r



Minimal costs:

$$\min_j \left(\delta d_{ij} + h_j\right)$$

Interactive apps: Barabasi-Albert (PhysRisk), Edge redirection (PhysRisk), Luck-and-reason (PhysRisk)

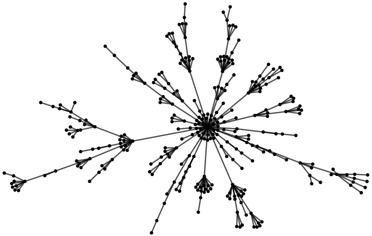
Continuum method

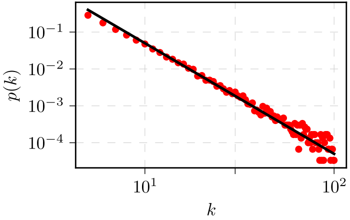
Expected degree of j-th node,

$$\frac{dk_j}{dt} = mp\left(t \to j\right) = m\frac{k_j}{\sum_m k_m},$$
$$\Rightarrow \quad k_j\left(t\right) \approx m\sqrt{\frac{t}{j}}.$$

Rearrangement gives us

$$j = N_{k_i > k} = \frac{m^2 t}{k^2}.$$





Delving deeper

Further general topics:

- Degree correlations
- Clustering
- Evolving networks
- Centrality and influence
- Strategic network formation
- Communities and their detection

Recent research directions:

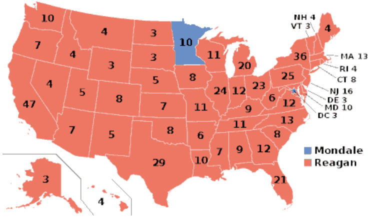
- Temporal evolution
- Multi-layer networks
- Hyper-graphs
- Higher-order networks
- Predicting missing edges
- Reconstructing processes

Recommendation: Barabasi "Network Science", "Social and Economic Networks" course (Coursera and Youtube)

Opinion dynamics

Topic, not a tool

- Elections, polls, census data
- Online discussion
- Collective behavior
- Laboratory experiments



Images: Gizmodo, Wikimedia, Wikimedia

Different kinds of models

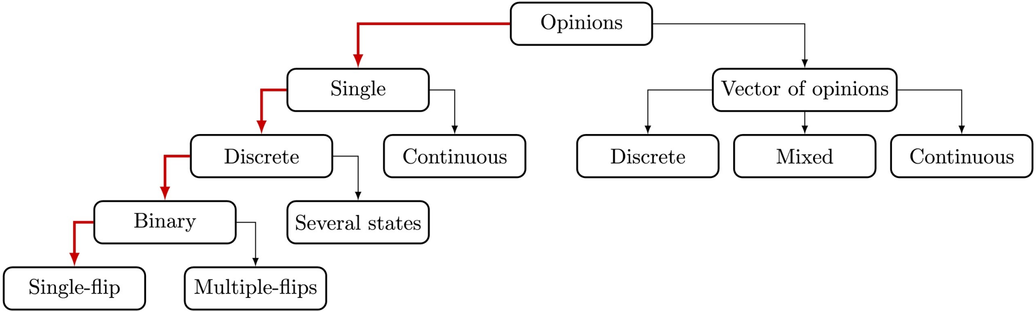
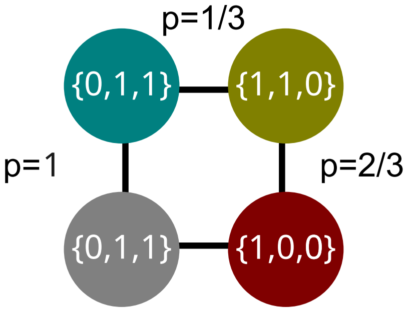


Figure: [Jedrzejewski and Sznajd-Weron (2019)]

Opinion vector: Axelrod model

- Opinion is given by d-dimensional vector.
- Each component may take q distinct values.

1. Choose a random agent i.
2. Choose a random neighbor j.
3. Interaction probability is proportional to the number of shared components.
4. During interaction i copies a single component from j.



Article: [Axelrod (1997)]; Interactive app: Physics of Risk

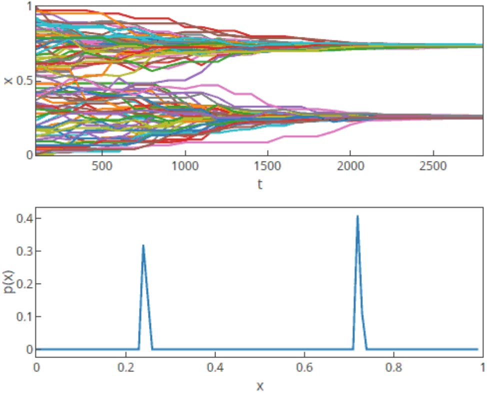
Continuous opinions: bounded confidence models

- Agents have continuous opinion x_i.
- Interactions between i and j are occur only if

$$|x_j(t) - x_i(t)| < \varepsilon.$$

- During interaction

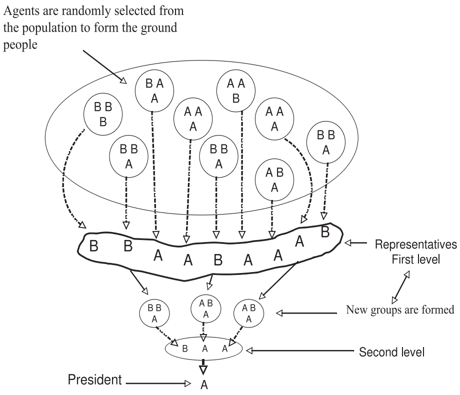
$$x_i(t+1) = x_i(t) + \mu \left[x_j(t) - x_i(t)\right].$$



Review: [Flache *et al.*(2017)]. Interactive app: Physics of Risk

Discrete opinion: Galam models

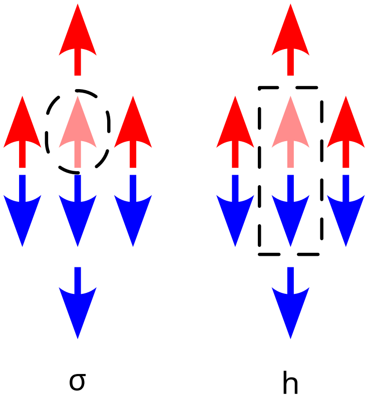
- Opinion is a discrete label
- Interactions occur in randomized groups:
  - All group members align with group's majority opinion
  - If group has no majority, then group members align with global minority.



Image/review: [Galam (2008)]. Interactive app: Physics of Risk

Noisy voter model

- Discrete (often) binary opinions
- Agents may change their opinion independently
- Agents may change their opinion by imitating their peers
- Interactions may occur on a complete network or another arbitrary social network



Reviews: [Redner (2019); Jedrzejewski and Sznajd-Weron (2019)]. Interactive apps: #voter-model (PhysRisk)

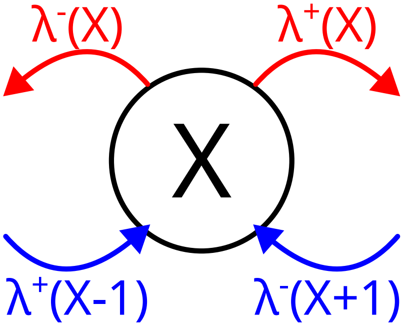
It is a birth-death process

We can formalize NVM using birth and death rates:

$$\lambda^+(X) = (N - X)\left[\sigma^+ + h\frac{X}{N^\alpha}\right], \quad \lambda^-(X) = X\left[\sigma^- + h\frac{N-X}{N^\alpha}\right].$$

Master equation:

$$\frac{\Delta p(X,t)}{\Delta t} = -\lambda^+(X)p(X,t) - \lambda^-(X)p(X,t) + \\ + \lambda^+(X-1)p(X-1,t) + \\ + \lambda^-(X+1)p(X+1,t).$$



Book: [van Kampen (2007)]

Thermodynamic ($N \to \infty$) limit

Rewrite rates:

$$\lambda_s^+(x) = N^2 \cdot (1-x)\left[\frac{\varepsilon^+}{N} + \frac{x}{N^\alpha}\right], \quad \lambda_s^-(x) = N^2 \cdot x\left[\frac{\varepsilon^-}{N} + \frac{1-x}{N^\alpha}\right].$$

Master equation:

$$\begin{aligned}\frac{\Delta p(x,t_s)}{\Delta t_s} = &- \lambda_s^+(x)p(x,t_s) - \lambda_s^-(x)p(x,t_s) \\ &+ \lambda_s^+(x-\Delta x)p(x-\Delta x,t_s) + \lambda_s^-(x+\Delta x)p(x+\Delta x,t_s) = \\ &= \left(\mathbf{E}^+ - 1\right)\left[\lambda_s^-(x)p(x,t_s)\right] + \left(\mathbf{E}^- - 1\right)\left[\lambda_s^+(x)p(x,t_s)\right].\end{aligned}$$

Here: $\mathbf{E}^\pm f(x) = f(x \pm \Delta x) \approx f(x) \pm \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x)$.

Book: [van Kampen (2007)]

Fokker-Planck equation

$$\frac{\partial p(x,t_s)}{\partial t_s} \approx -\frac{\partial}{\partial x}\left[\frac{\lambda_s^+(x)-\lambda_s^-(x)}{N}p(x,t_s)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\frac{\lambda_s^+(x)+\lambda_s^-(x)}{N^2}p(x,t_s)\right] \approx$$

$$\approx -\frac{\partial}{\partial x}\left[\left\{\varepsilon^+(1-x)-\varepsilon^- x\right\}p(x,t_s)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\frac{2x(1-x)}{N^\alpha}p(x,t_s)\right].$$

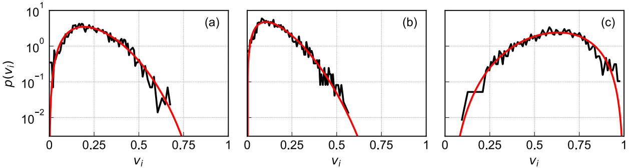
Stationary distribution (with $\alpha = 0$):

$$0 = -\left\{\varepsilon^+(1-x)-\varepsilon^- x\right\}p_{st}(x) + \frac{d}{dx}\left[x(1-x)\,p_{st}(x)\right] \quad \Rightarrow$$

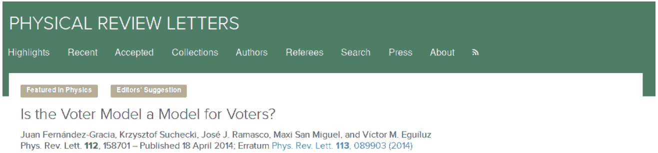
$$p_{st}(x) = C_N \cdot x^{\varepsilon^+-1}(1-x)^{\varepsilon^--1}.$$

Book: [Risken (1996)]

Beta distribution fits empirical data



Party (SK (a), LKDP (b) and LDDP (c)) vote shares in Lithuanian 1992 parliamentary election.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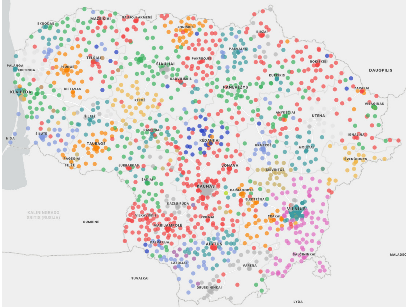
Is the Voter Model a Model for Voters?

Juan Fernández-Gracia, Krzysztof Suchecki, José J. Ramasco, Maxi San Miguel, and Víctor M. Eguíluz
Phys. Rev. Lett. **112**, 158701 – Published 18 April 2014; Erratum Phys. Rev. Lett. **113**, 089903 (2014)



Lithuanian 2022 municipality election results map.

Figure: [Kononovicius (2018)]; Image: rinkimai.maps.lt; Article: [Fernandez-Gracia *et al.*(2014)].

Delving deeper

- q-voter model
- Multi-state voter model
- Non-Markovian dynamics
- Polarization (physicsworld.com)
- Dynamics on networks
- Compatibility with social sciences

Recommendations: [Castelano *et al.*(2009); Flache *et al.*(2017); Peralta *et al.*(2023)]; Foreground image: source lost; Background image: "spinson".

Thank you!

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