Income Distributions: Models and empirical studies using UK data

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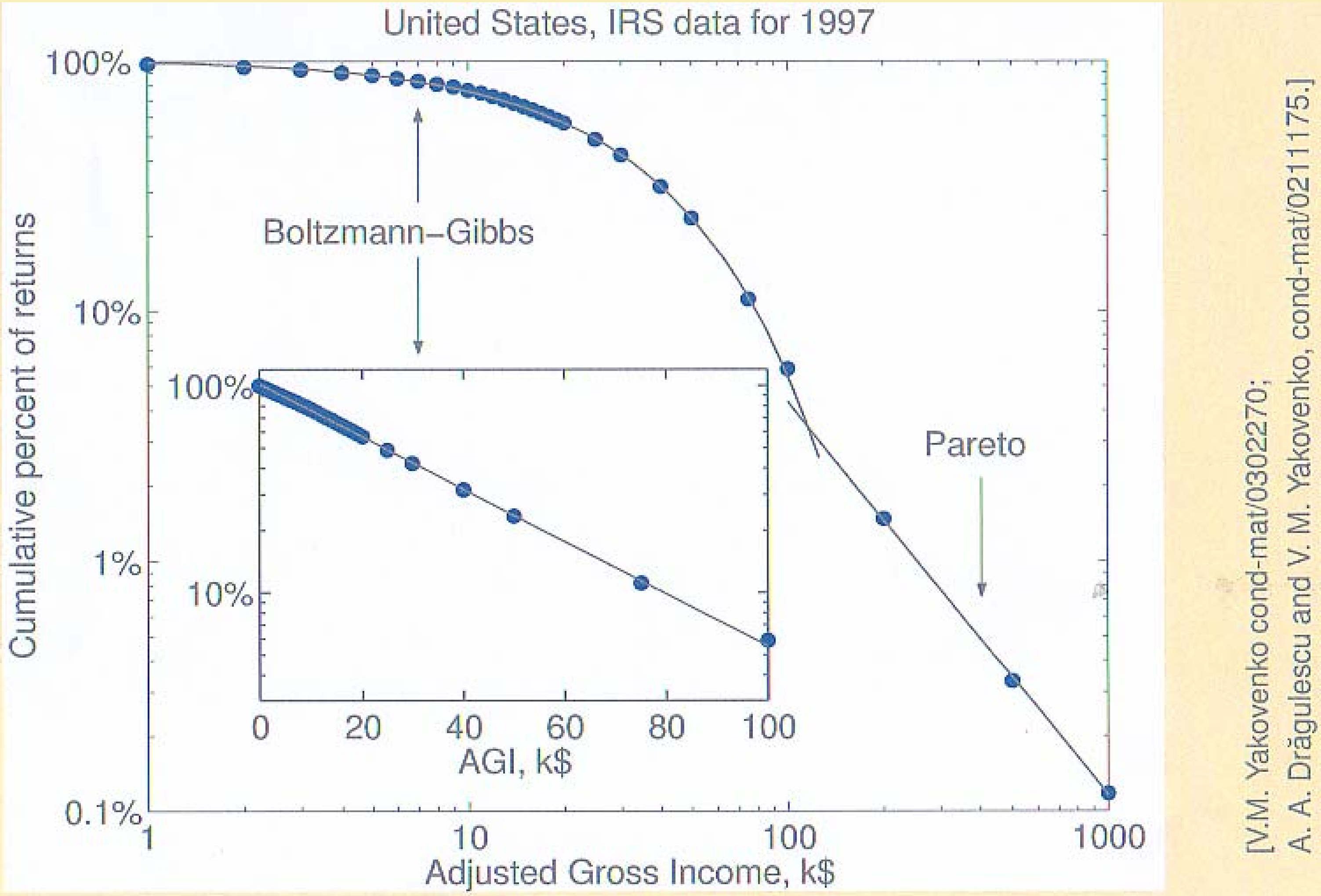
Summary

- Pareto exponents
- Models
- UK Data
  - the poor and middle class (you and me)
  - from millionaires to billionaires

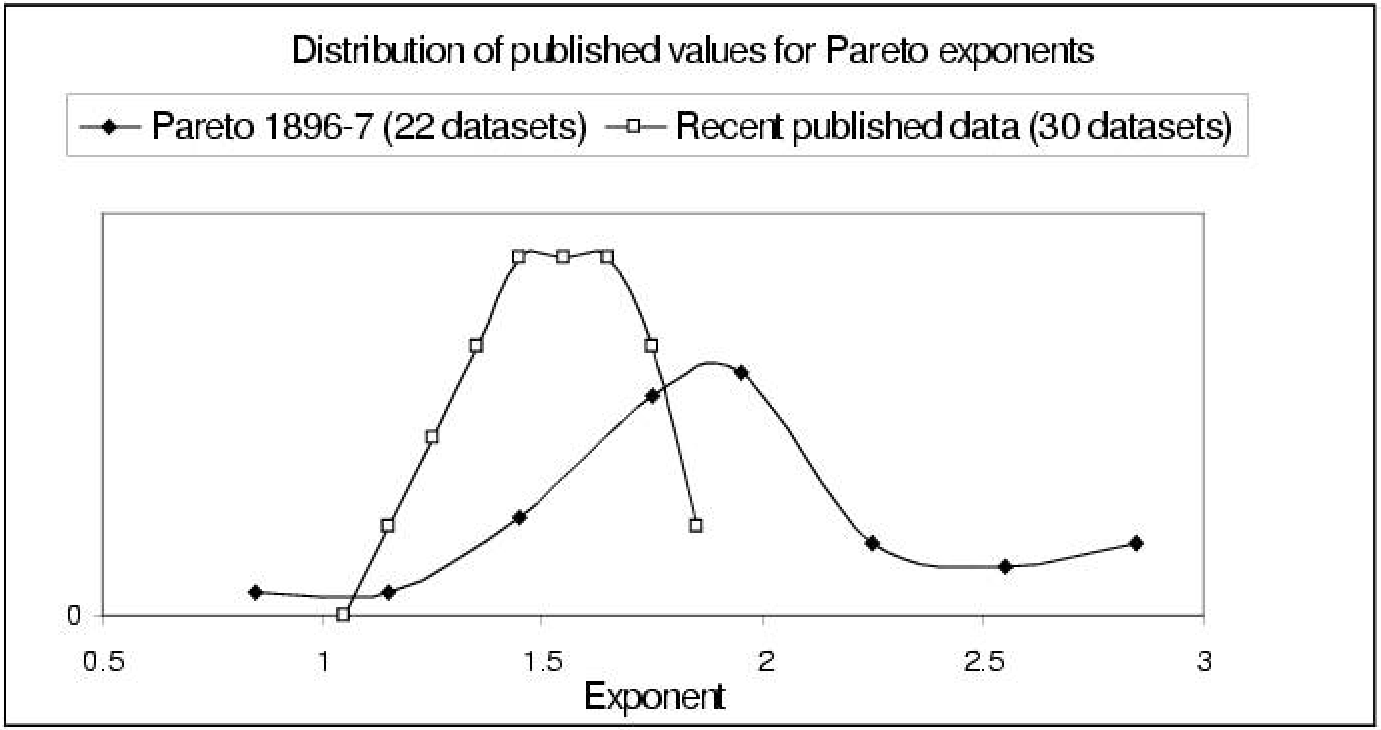


Exponential distribution at low incomes, Pareto at high incomes

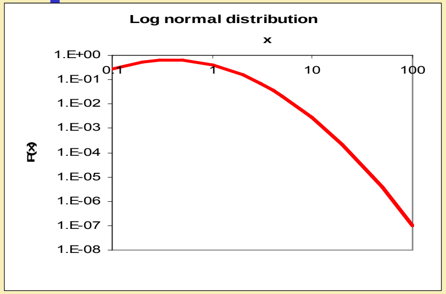
“This law for the distribution of wealth ...... can be compared to Kepler’s law in astronomy..”

Country	Year	α	Country	Year	α
England	1843	1.50	Perouse, village		1.69
	1879-80	1.35	Perouse, campagne		1.37
Prussia	1852	1.89	Ancone, Arezzo, Parme and Pise		1.32
	1876	1.72	Italian villages		1.45
	1881	1.73	Basle	1887	1.24
	1886	1.68	Paris		1.57
	1890	1.60	Augsburg	1471	1.43
	1894	1.60		1498	1.47
Saxony	1880	1.58		1512	1.28
	1886	1.51		1526	1.13
Florence		1.41	Peru	end of 18th century	1.79

Country	Source	Distributions	Exponents
Egypt	S.H. (14^{th}B.C.)	Par.	$\alpha = 1.59 \pm 0.19$
Japan	I.T. (1992)	Par.	$\alpha = 2.057 \pm 0.005$
	I. (1998)	Par.	$\alpha = 1.98$
	I.T. (1998)	Par.	$\alpha = 2.05$
	I. / I.T. (1998)	Par.	$\alpha = 2.06$
	I. (1887-2000)	LN / Par.	$\overline{\alpha} \sim 2.0$
U.S.A.	I.T. (1997)	Par.	$\alpha = 1.6$
Japan	I.T. (2000)	Par.	$\alpha = 2.0$
U.S.A.	I. (1992-1998)	Exp. / Par.	
	I. (1998)	Exp. / Par.	$\alpha = 1.7 \pm 0.1$
U.K.	Inhe. T. (1996)	Exp. / Par.	$\alpha = 1.9$
Italy	I. (1977-2002)	LN / Par.	
	I. (1987)	Par.	$\alpha = 2.09 \pm 0.002$
	I. (1993)	LN / Par.	$\alpha = 2.74 \pm 0.002$
	I. (1998)	Par.	$\alpha = 2.76 \pm 0.002$
Australia	I. (1993-97)	Par.	$\alpha \sim 2.2\text{-}2.6$
U.S.A.	I. (1997)	D. Par.	$\alpha = 22.43$ / $\beta = 1.43$
Canada	I. (1996)	D. Par.	$\alpha = 4.16$ / $\beta = 0.79$
Sri-Lanka	I. (1981)	D. Par.	$\alpha = 2.09$ / $\beta = 3.09$
Bohemia	I. (1933)	D. Par.	$\alpha = 2.15$ / $\beta = 8.40$
U.S.A.	1980	Par.	$\alpha = 2.2$
	1989	Par.	$\alpha = 1.63$
	2001	G. / Par.	
U.K.	1996	Par.	$\alpha = 1.85$
	1998-99	Par.	$\alpha = 1.85$
U.S.A.	I. (1992)	Exp. / LN	
U.K.	I. (1992-2002)	Exp. / LN	
India	W. (2002-2004)	Par.	$\alpha \sim 0.81\text{-}0.92$
	I. (1997)	Par.	$\alpha = 1.51$
U.S.A.	W. (1996)	Par.	$\alpha = 1.36$
	W. (1997)	Par.	$\alpha = 1.35$
U.K.	W. (1970)	Par.	
	W. (1997)	Par.	$\alpha = 1.06$
Sweden	W. (1965)	Par.	$\alpha = 1.66$
France	W. (1994)	Par.	$\alpha = 1.83$
U.K.	Inhe. T. (2001)	Par.	$\alpha = 1.78$
Portugal	I.T. (1998-2000)	Par.	$\alpha \sim 2.30\text{-}2.46$


Distribution of published values for Pareto exponents
Pareto 1896-7 (22 datasets)
Recent published data (30 datasets)
0
0.5
1
1.5
2
2.5
3
Exponent

Multiplicative random processes



- Gibrat $x_{t+1} = (1+\xi_t)x_t \Rightarrow P(x) = \frac{e^{-\frac{\ln^2 x}{2}}}{x\sqrt{2\pi}}$
- Levy & Solomon
  - Int J Mod Phys C7 1996 65-72

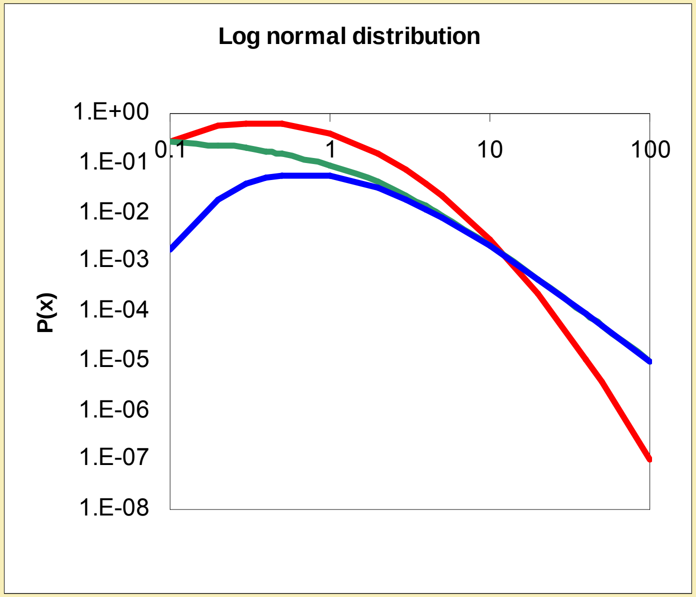
$$x_{t+1} - x_t = \xi_t x_t + B_t \Rightarrow P(\ln x) = e^{-\alpha \ln x}$$

$$\Rightarrow P(x) \sim x^{-(1+\alpha)}$$

- Solomon & Richmond
  - Int J Mod Phys C12 2001 1-11

$$x_{t+1} - x_t = (\xi_i - \overline{\xi} - a)x_t + a$$

$$\Rightarrow P(x) = \frac{1}{\mathrm{N}} \frac{\exp[-\frac{a}{Dx}]}{x^{2+\frac{a}{D}}}$$



Lotka-Volterra models

Solomon S et al, Theoretical analysis and simulations of the generalized Lotka-Volterra model, Phys. Rev. E **66**, n 3 1, 031102/1-031102/4, (2002); Solomon S, Richmond P, Stable power laws in variable economies; Lotka-Volterra implies Pareto-Zipf, Eur. Phys. J. B **27**, n 2, p 257-261, (2002); Richmond P, Solomon S, Power laws are disguised Boltzmann laws Int. J. Mod. Phys. C **12**, n 3, p 333-43, (2001)

- Basic idea, N agents, wealth m
- Random multiplicative wealth and wealth redistribution

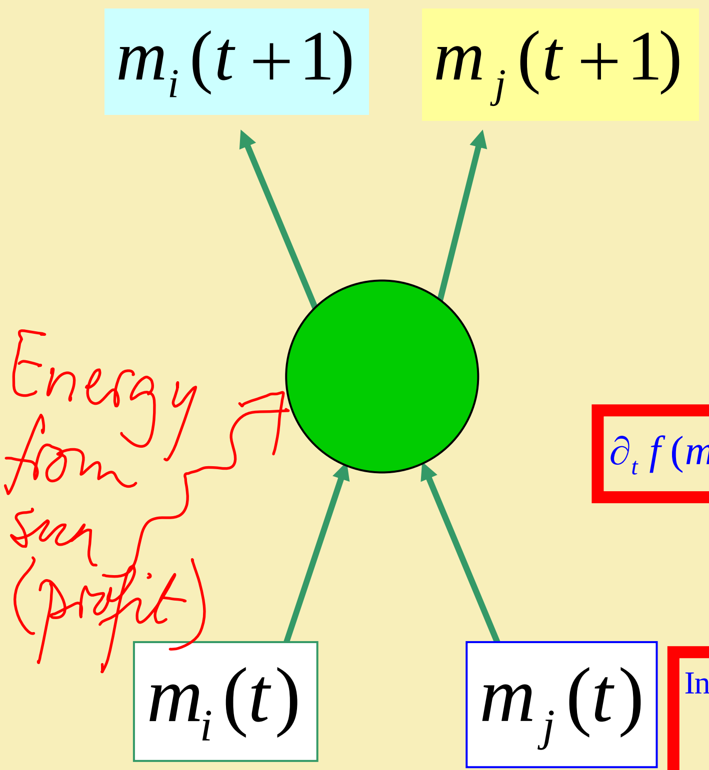
$$m_i(t+1) = e^{\xi(t)}\left[(1-a)m_i + \frac{a}{N}\sum_j m_i\right] \sim m_i(1+\xi(t)-c) + a\overline{m}; \quad \overline{m} = \frac{a}{N}\sum_j m_i$$

- Mean field $\Sigma m_i / N \to \langle m \rangle$ Rescale wealth $x_i \to m_i / \langle m \rangle$
- Equations decouple

$$P(x) \sim \frac{1}{N}\frac{\exp(-(\alpha-1)/x)}{x^{1+\alpha}}; \quad \alpha = 1 + a/D$$

Money in gas like markets

F Slanina Phys Rev E69 (2004) 46102-1-7



molecules → agents
Scattering → money exchange
β fraction exchanged
ε average profit

$$\begin{pmatrix} m_{i,t+1} \\ m_{j,t+1} \end{pmatrix} = \begin{pmatrix} 1+\varepsilon-\beta & \beta \\ \beta & 1+\varepsilon-\beta \end{pmatrix} \begin{pmatrix} m_i \\ m_j \end{pmatrix}$$

$$m_i(t+1) + m_j(t+1) > m_i(t) + m_j(t)$$

$$\partial_t f(m) + f(m) = \int f(m_1) f(m_2) \delta((1-\beta+\varepsilon)m_1 + \beta m_2 - m) dm_1 dm_2$$

In limit $\beta \to 0, \varepsilon \to 0, \alpha$ fixed: $\beta = (\alpha-1)\varepsilon^2/2 + O(\varepsilon^3)$

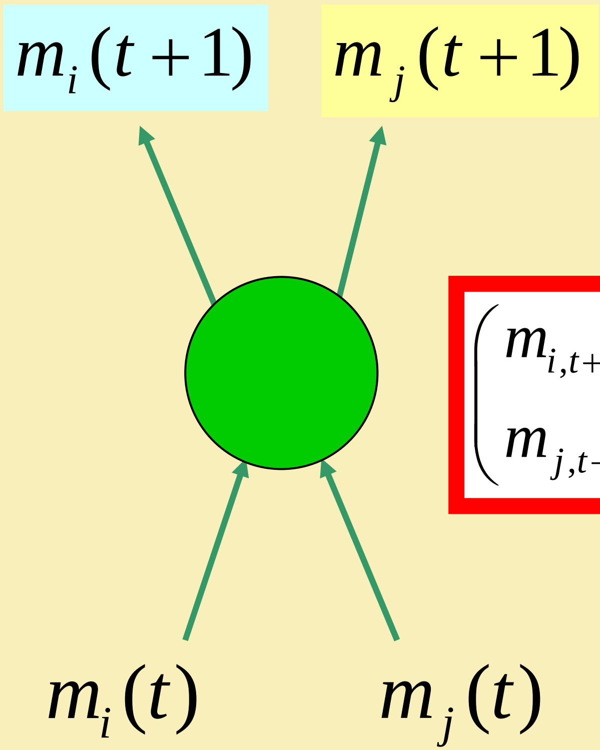
$$\hat{\Phi}(p) \sim p^{\alpha/2} K_\alpha(2\sqrt{p(\alpha-1)})$$

$$\Phi(x) = C \frac{e^{-(\alpha-1)/x}}{x^{1+\alpha}}$$

$$C = \frac{(\alpha-1)^\alpha}{\Gamma(\alpha)}$$

GLV!

Money in gas like markets with random exchange



Fraction saved

Money from interaction
Speculation/ competition
$0 \le \varepsilon \le 1$

$$\begin{pmatrix} m_{i,t+1} \\ m_{j,t+1} \end{pmatrix} = \begin{pmatrix} \lambda_i + \varepsilon(1-\lambda_i) & \varepsilon(1-\lambda_j) \\ (1\text{-}\varepsilon)(1-\lambda_i) & \lambda_j + (1-\varepsilon)(1-\lambda_j) \end{pmatrix} \begin{pmatrix} m_i \\ m_j \end{pmatrix}$$

$$m_i(t+1) + m_j(t+1) = m_i(t) + m_j(t)$$

Yields power law of unity -
but cf Repetowicz poster – exponential decay

United Kingdom Income Data
New Earning survey (NES)
National Statistics Office (www.statistics.gov.uk)

- Takes 1% sample of all employees in Great Britain
  - UK excluding Northern Ireland).
  - Includes full and part time employees.
  - Individuals identified through national insurance numbers and surveys of major employers.
  - Provides weekly income before tax
  - large, high quality sample
- Only includes wage earners,
  - Excludes unemployed, self-employed, those earning less than lowest tax threshold, those living on private income.
- Data used from 1992 to 2002 (inc)
- Use data points from £101-£110 to £1191-£1200. (Data above and below these levels lumped together, not included).

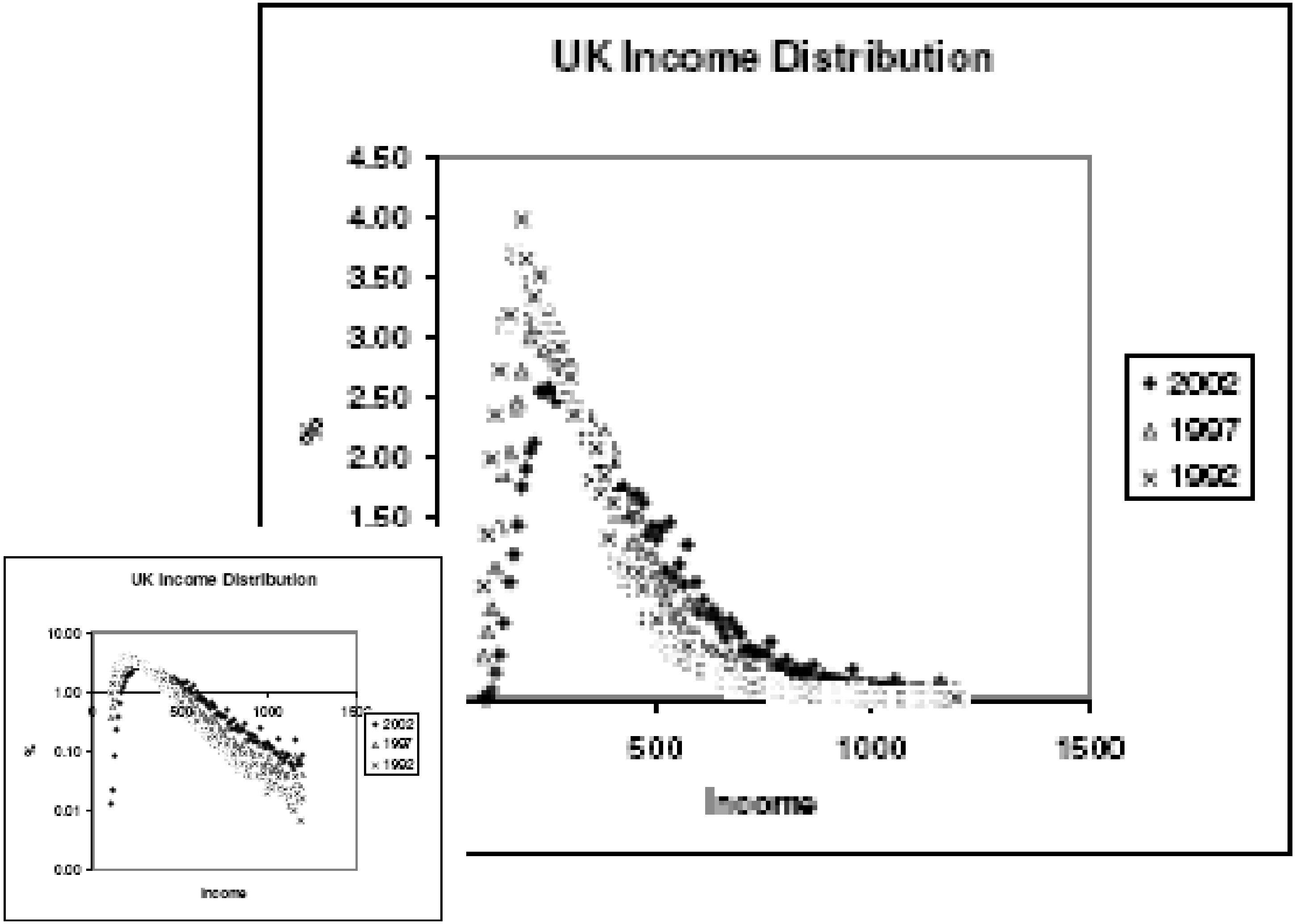


Fig. 1.7 Distribution functions for UK income for 2002, 1997 and 1992.

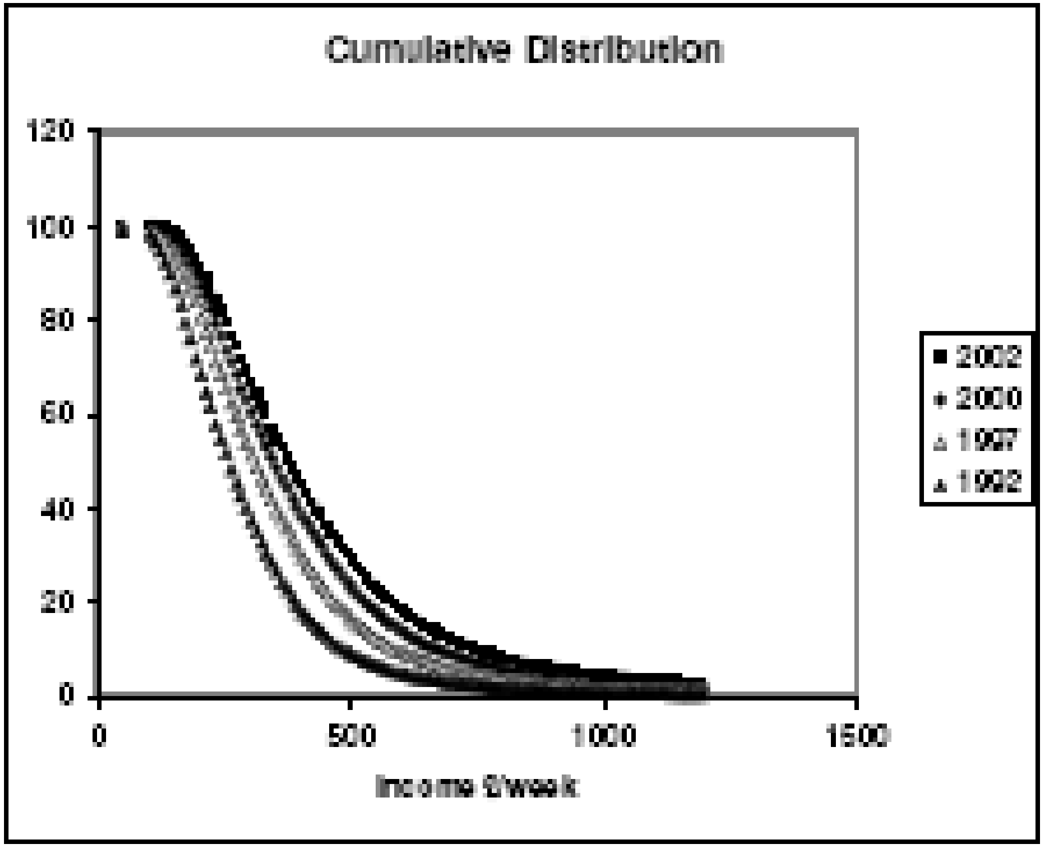


Fig. 1.9 Cumulative distribution functions for UK income over 1992, 1997, 2000 and 2002.

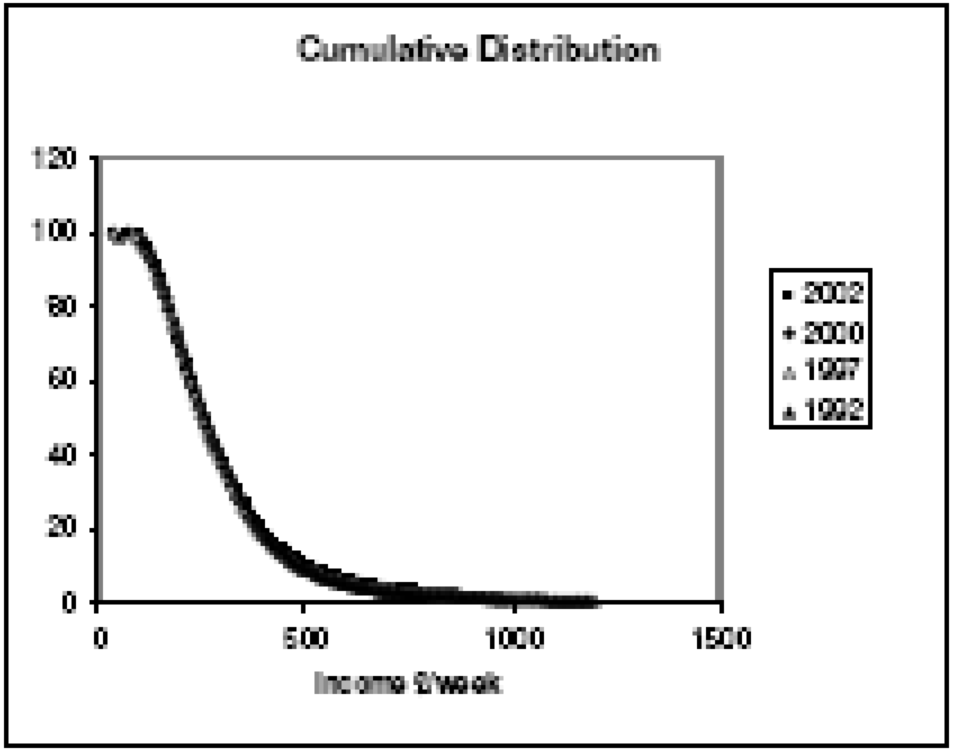
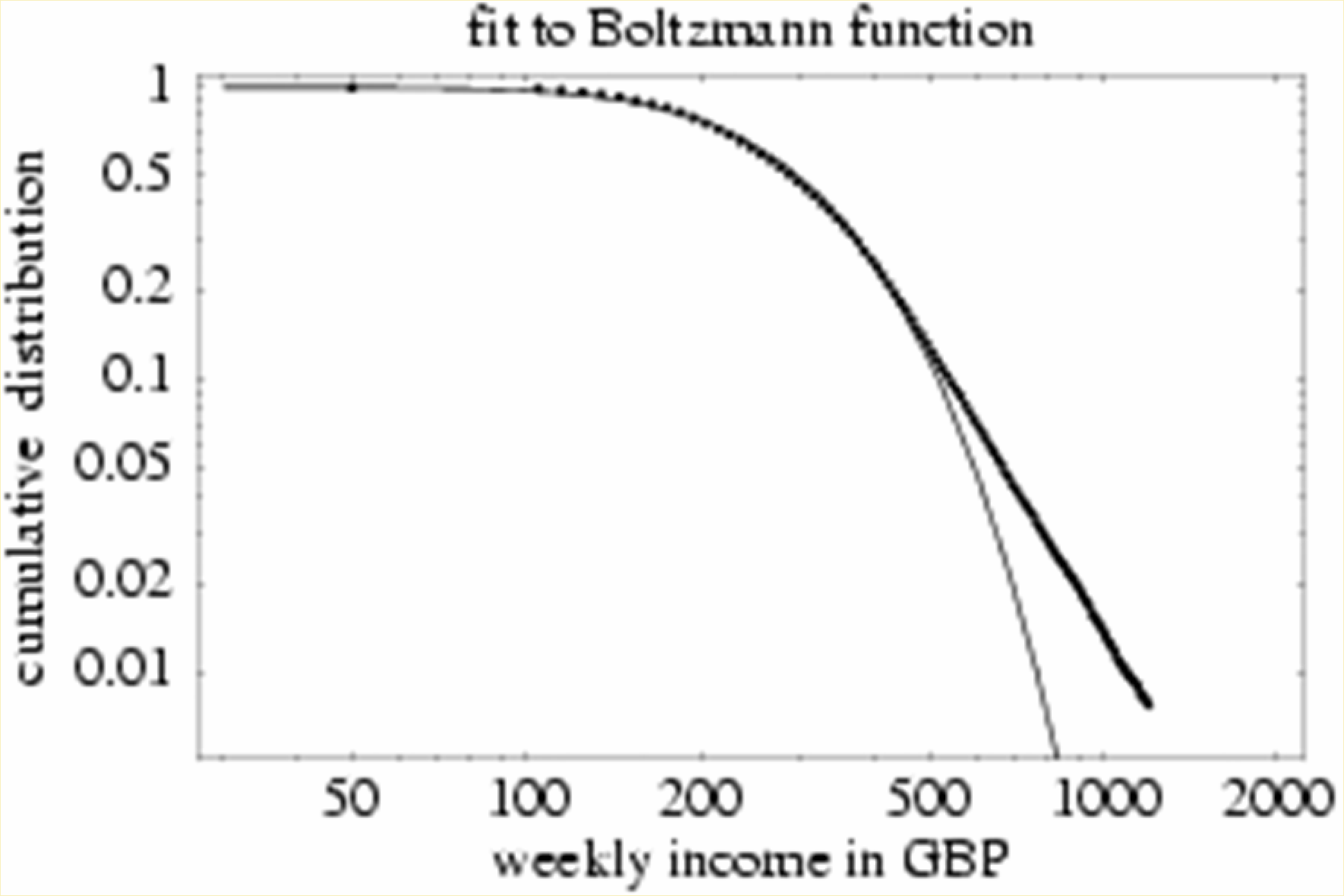
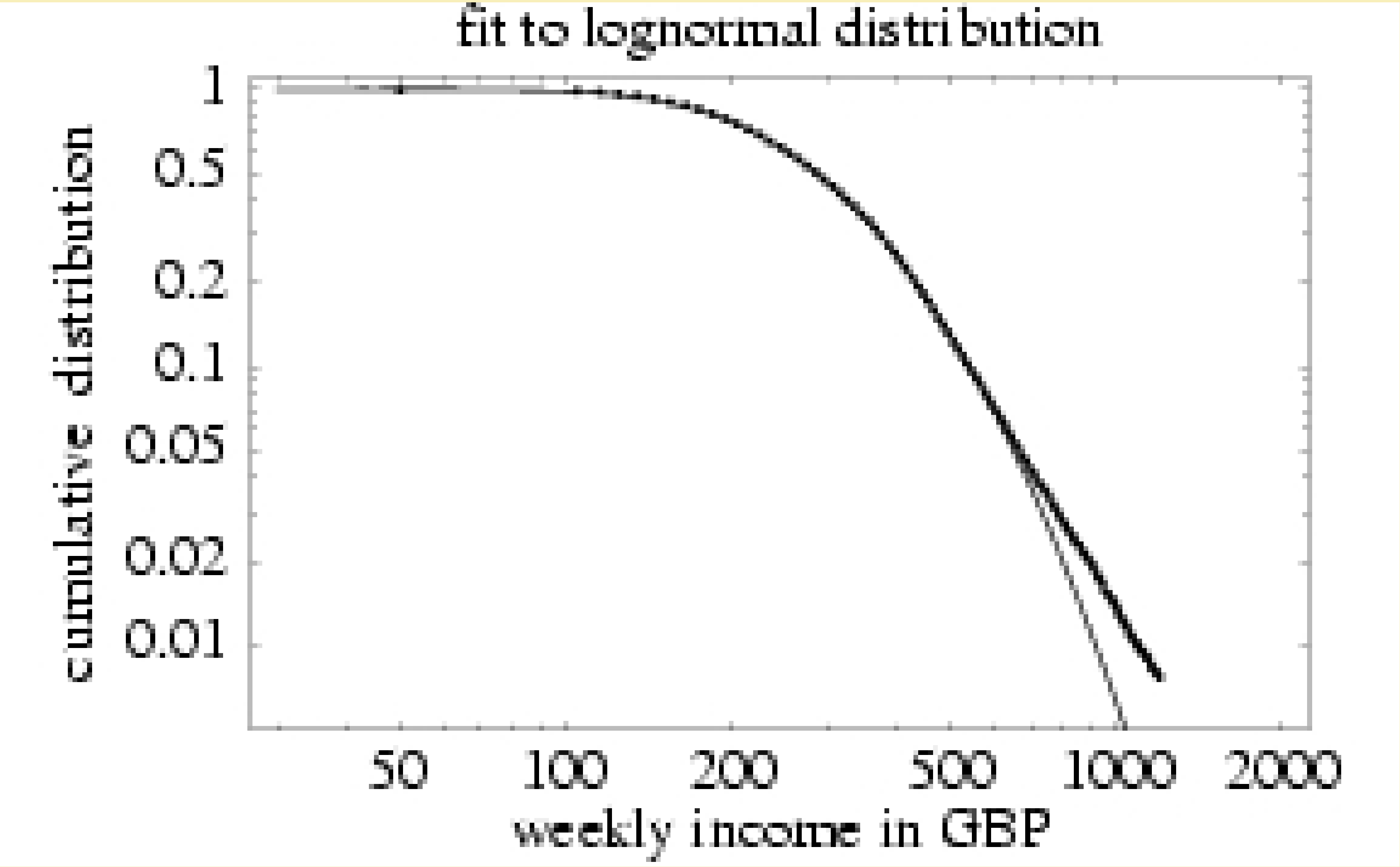


Fig. 1.10 UK income distributions for 1992 to 2002 scaled using a scaling factor for incomes of 4% per annum across the entire decade.


fit to Boltzmann function
1
0.5
0.2
0.1
0.05
0.02
0.01
cumulative distribution
50
100
200
500
1000
2000
weekly income in GBP


fit to lognormal distribution
1
0.5
0.2
0.1
0.05
0.02
0.01
cumulative distribution
50
100
200
500
1000
2000
weekly income in GBP

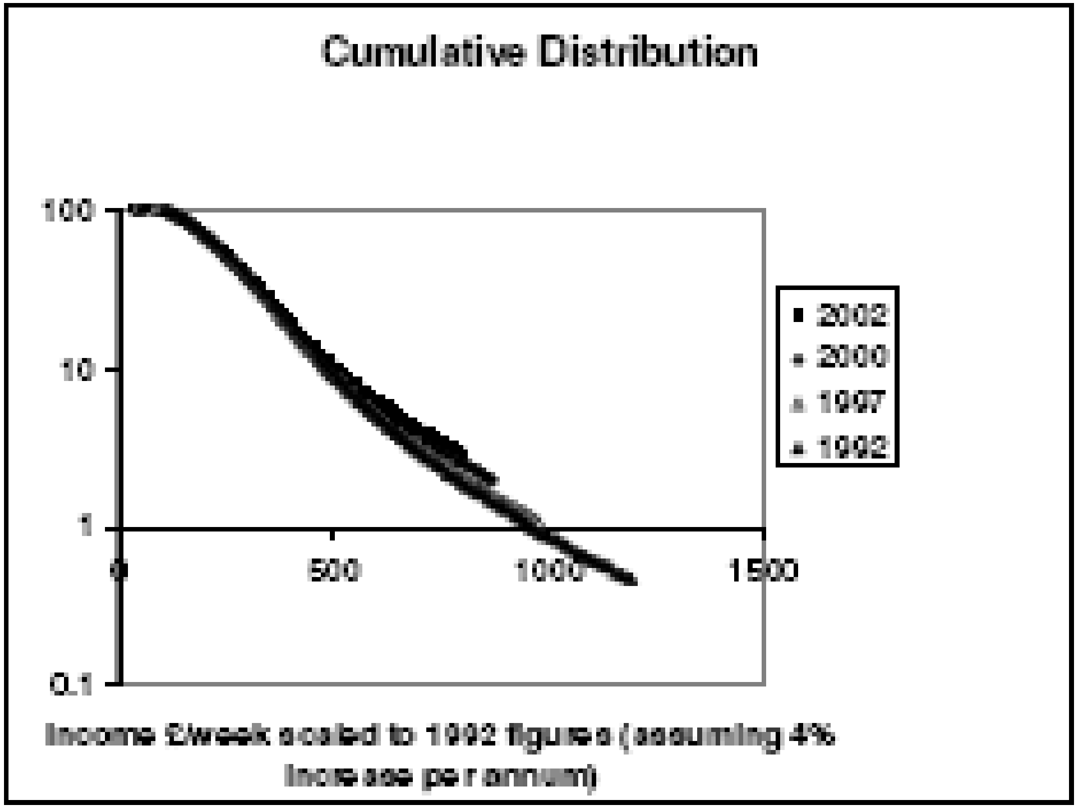


Fig. 1.12 *Scaled* income distributions for the years 2002, 2000, 1997 and 1992 plotted on a log linear scale.

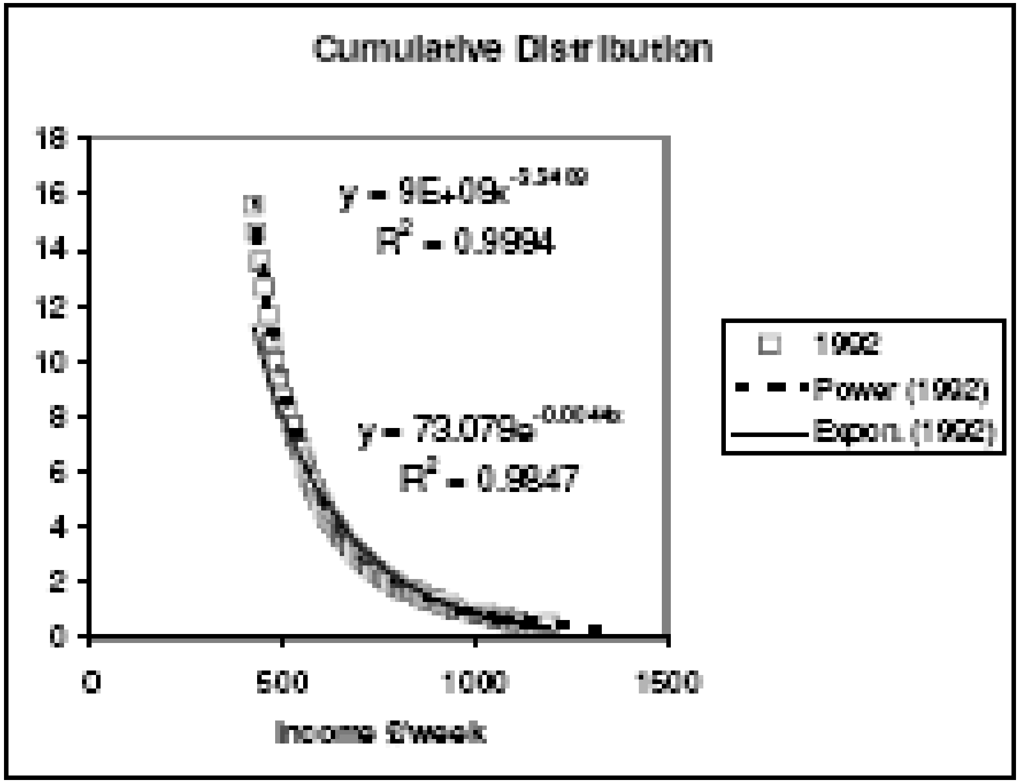


Fig. 1.13 The higher income part of the cumulative distribution for data for 1992 illustrating the comparison of power law and exponential fits.

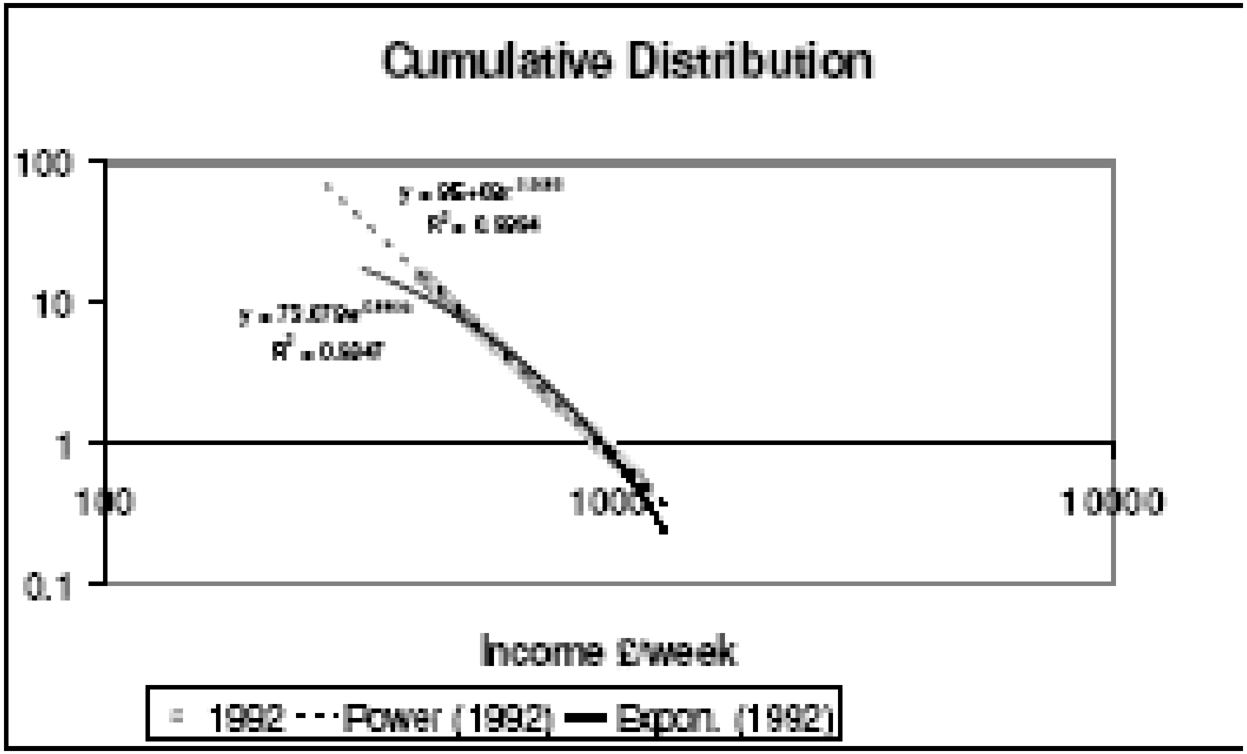


Fig. 1.14 Log-log plot of income data for 1992 showing trend lines for exponential and power laws.

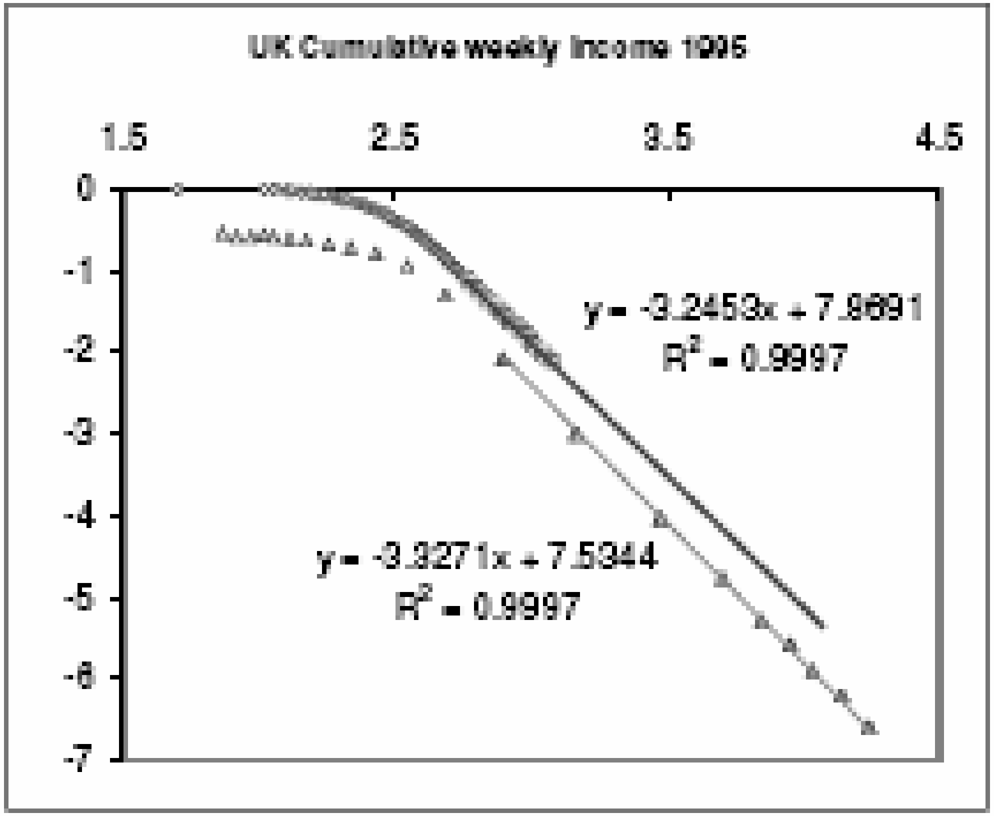


Fig. 1.15 NES income data (diamonds) for the UK together with points taken from the publication of Cranshaw (triangles) shown as a $log10 - log10$ plot.

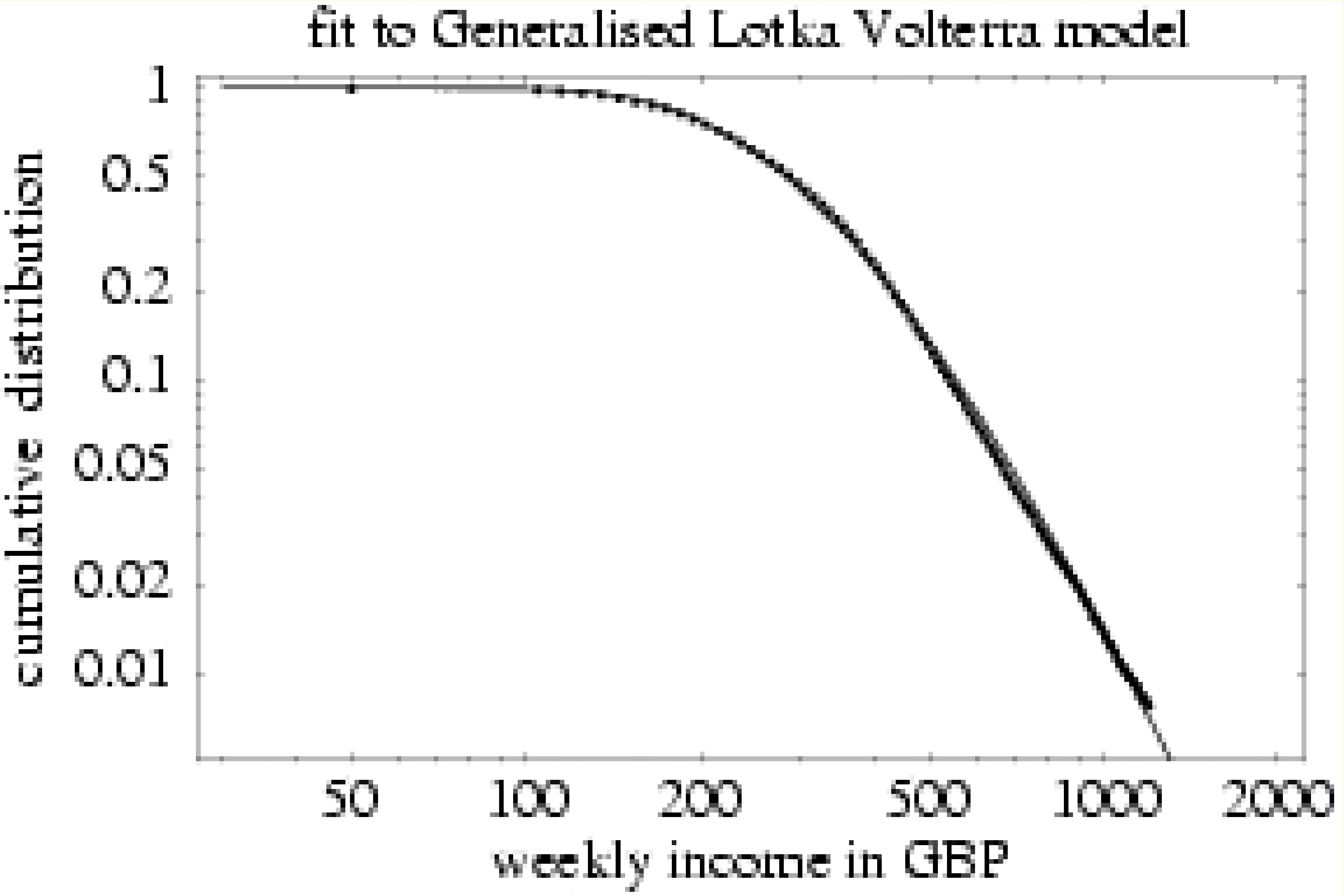
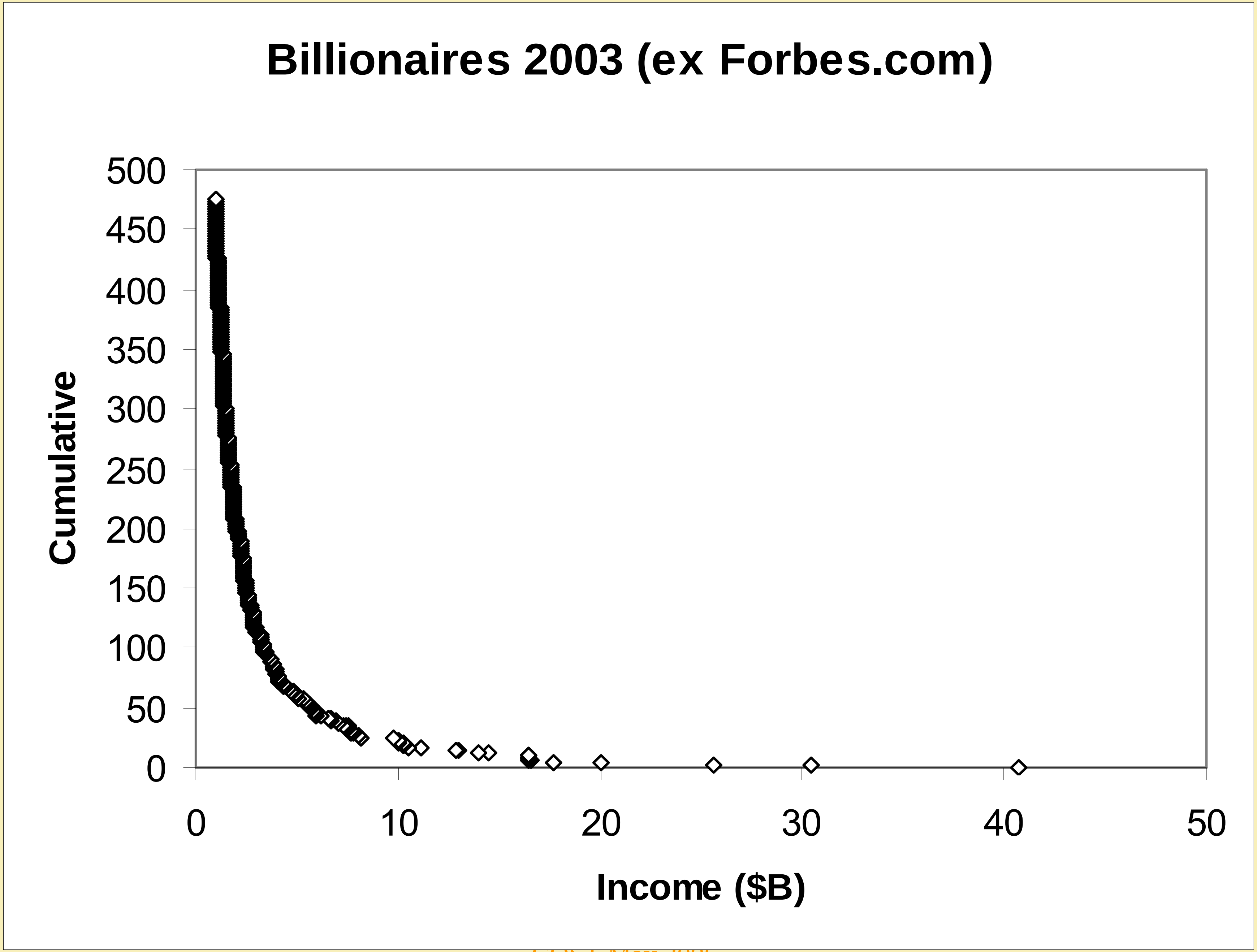
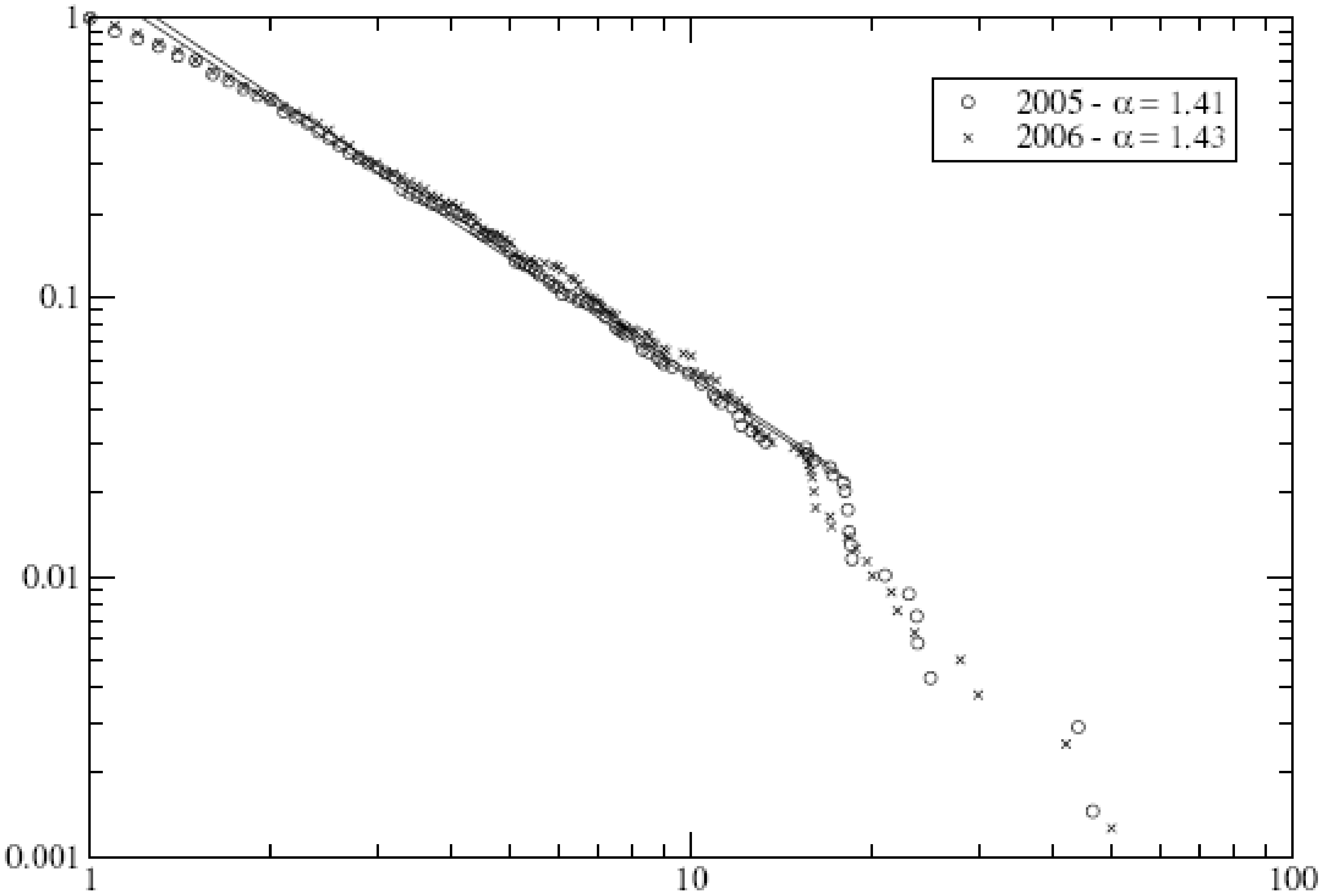

fit to Generalised Lotka Volterra model
cumulative distribution
1
0.5
0.2
0.1
0.05
0.02
0.01
50
100
200
500
1000
2000
weekly income in GBP

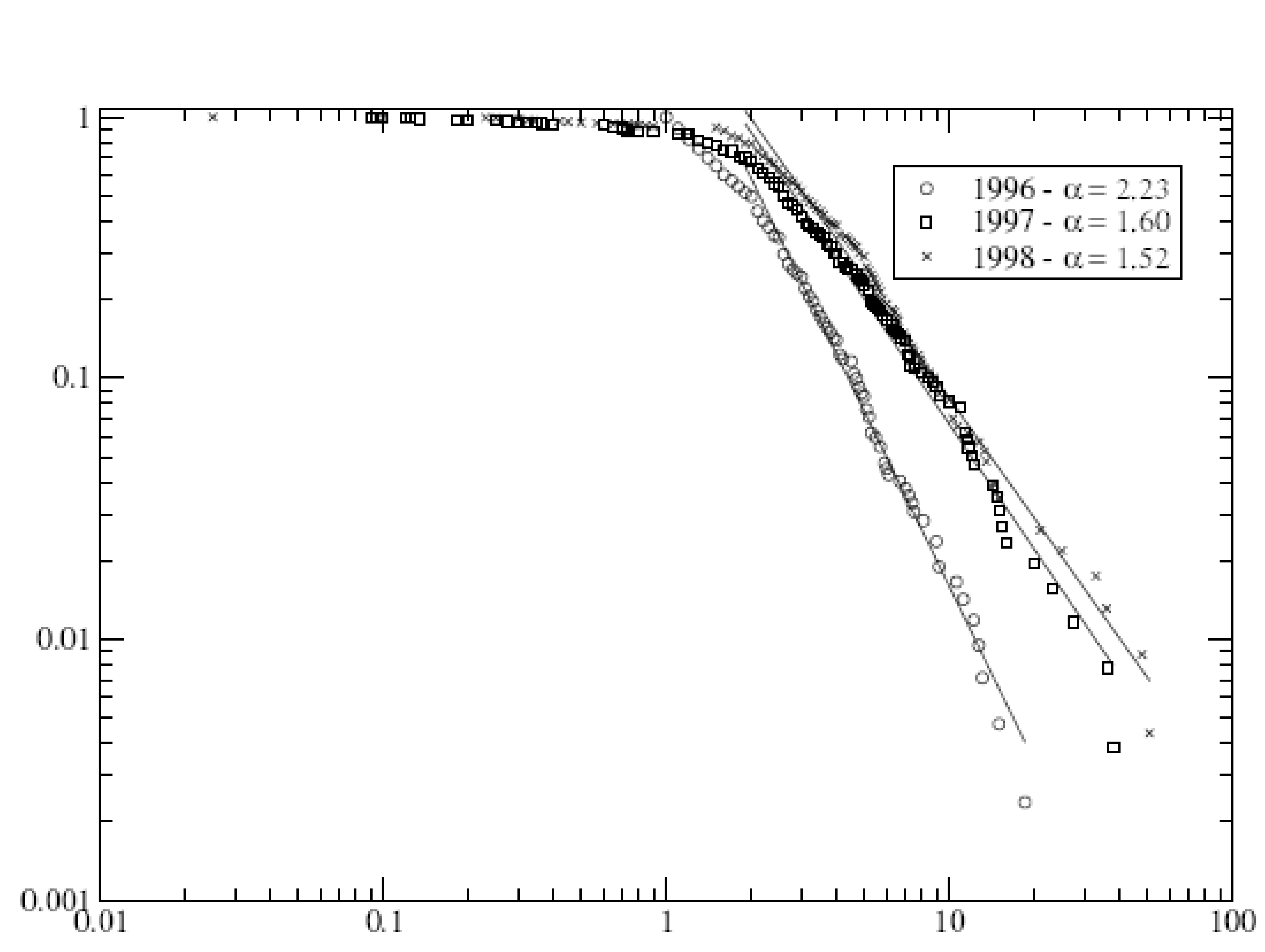
Table 1.3 Power laws for UK income data over the period 1992 to 2002.

Year	Power law Index	Year	Power law Index
1992	3.34	1998	3.15
1993	3.25	1999	3.15
1994	3.21	2000	3.15
1995	3.21	2001	2.68
1996	3.15	2002	2.70
1997	3.15		

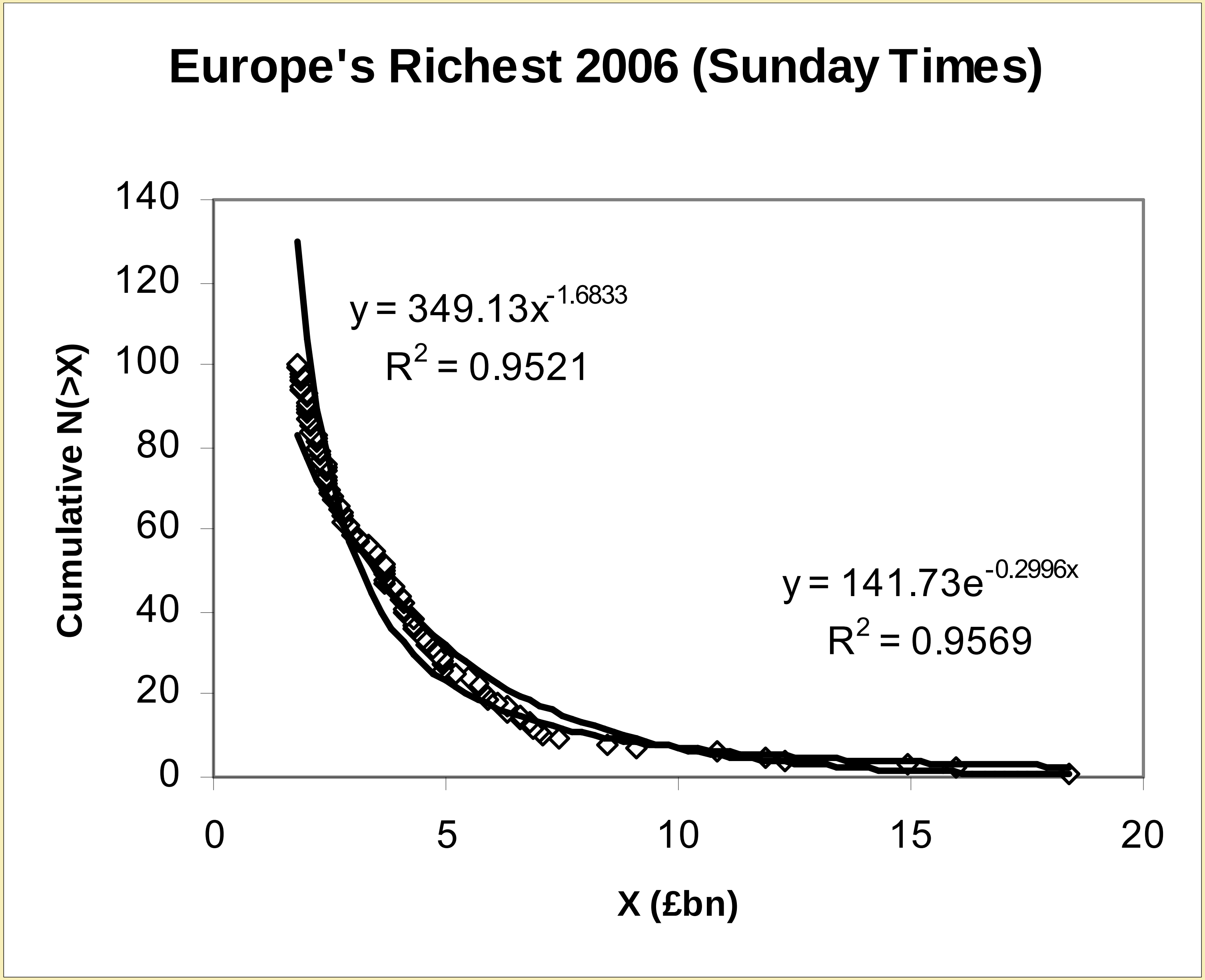

Billionaires 2003 (ex Forbes.com)
Cumulative
500
450
400
350
300
250
200
150
100
50
0
0
10
20
30
40
50
Income ($B)

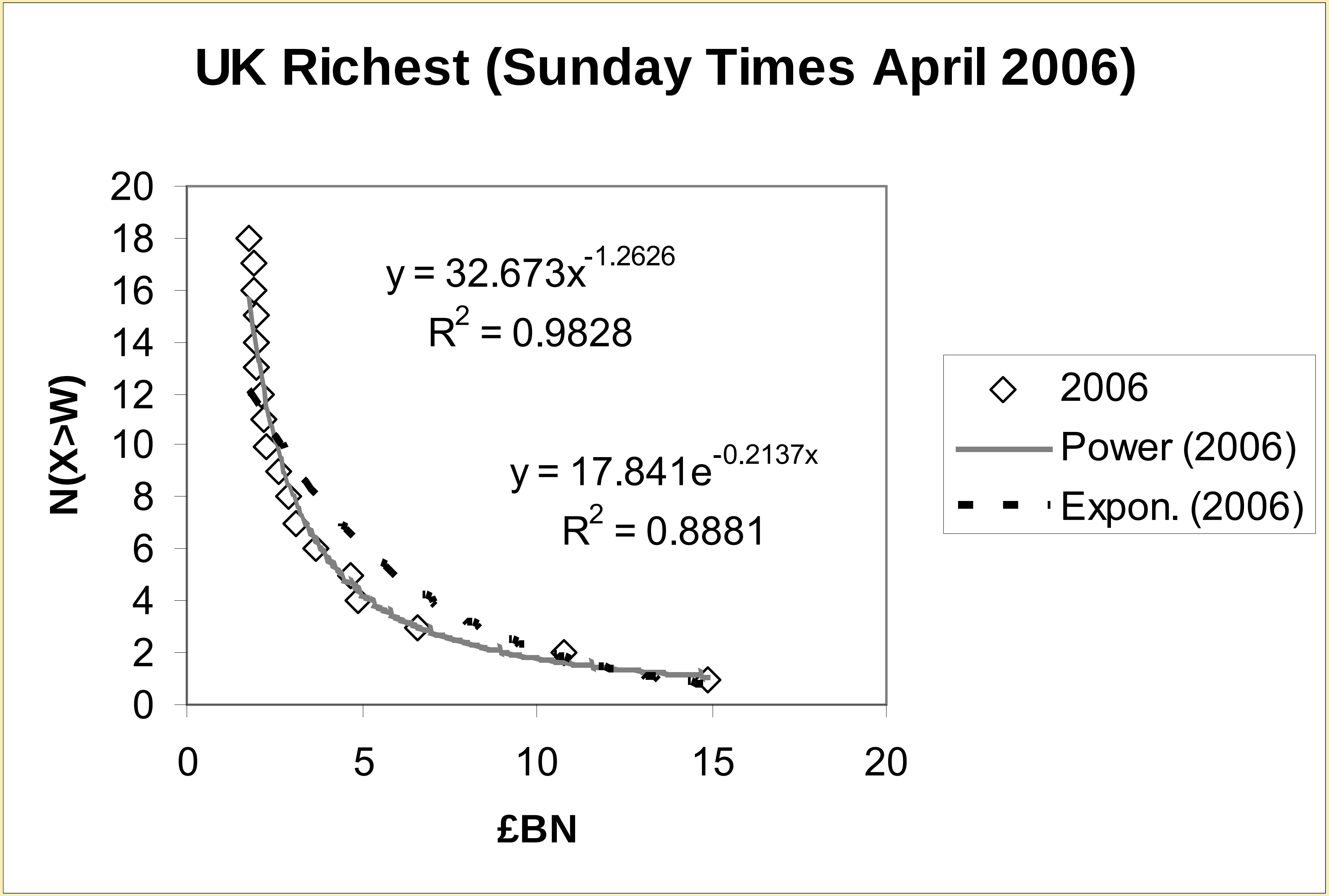


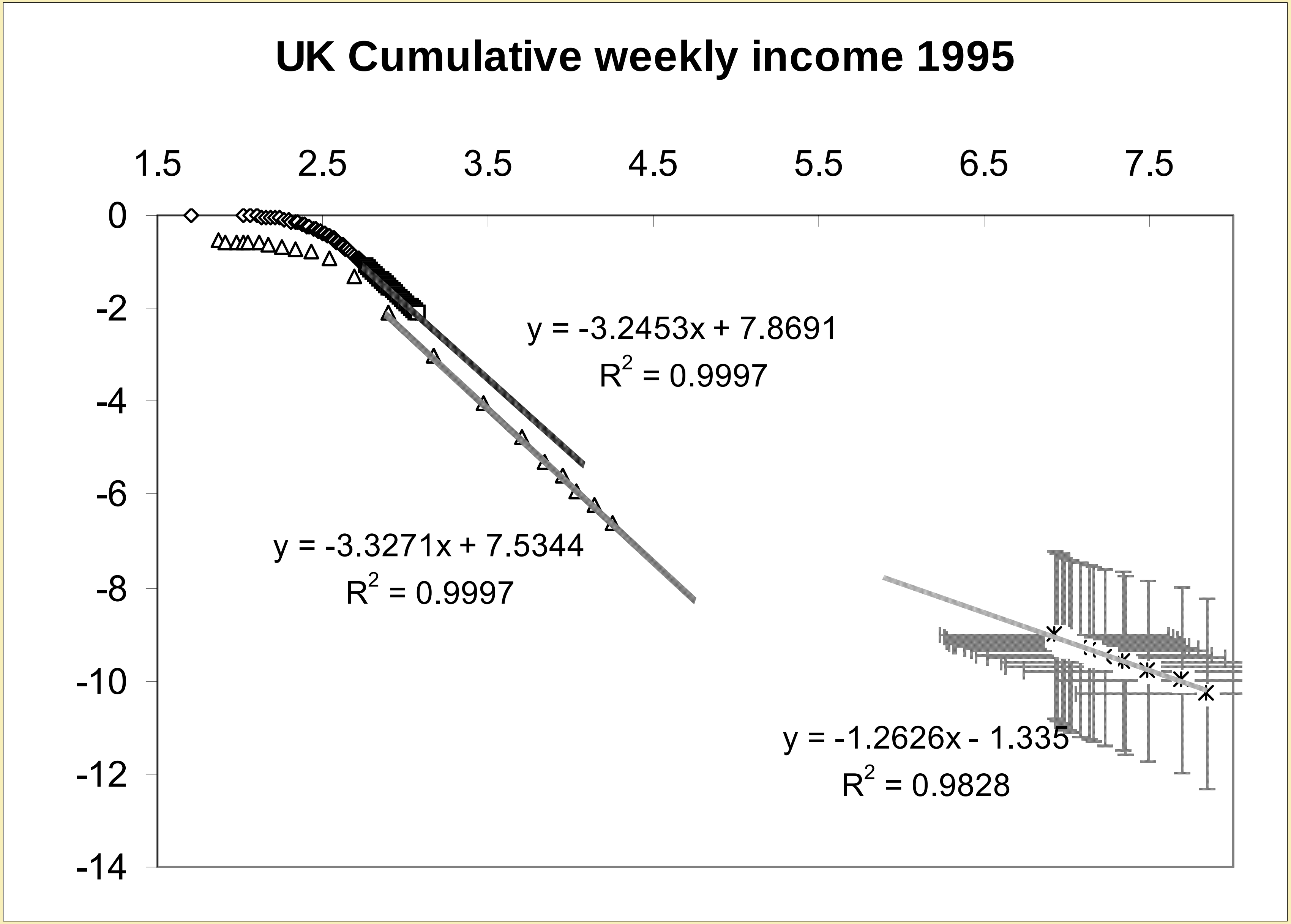
1
0.1
0.01
0.001
1
10
100
2005 - α = 1.41
2006 - α = 1.43

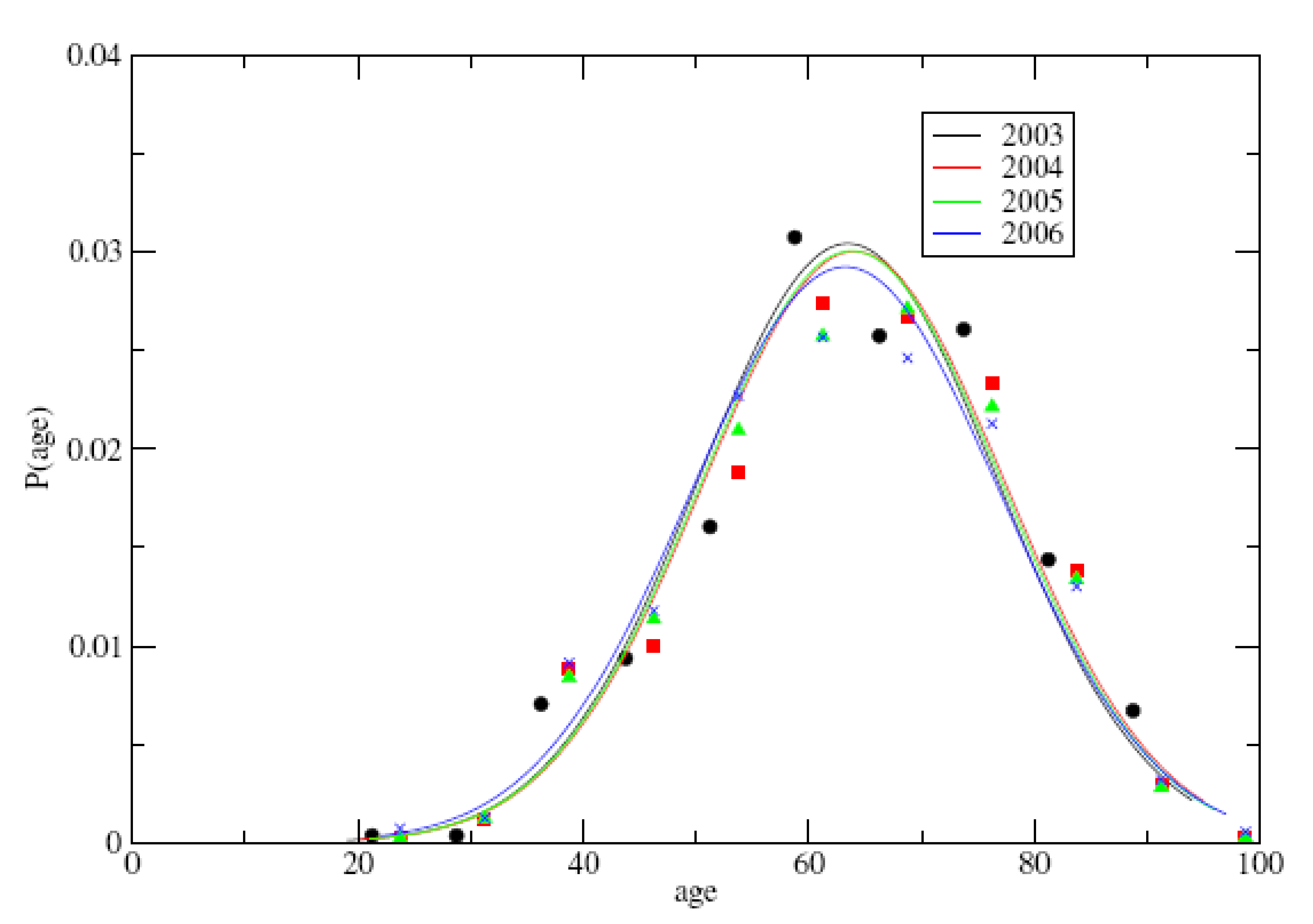


1
0.1
0.01
0.001
0.01
0.1
1
10
100
1996 - α = 2.23
1997 - α = 1.60
1998 - α = 1.52


Europe's Richest 2006 (Sunday Times)
y = 349.13x^-1.6833
R^2 = 0.9521
y = 141.73e^-0.2996x
R^2 = 0.9569
Cumulative N(>X)
X (£bn)
140
120
100
80
60
40
20
0
0
5
10
15
20


UK Richest (Sunday Times April 2006)
y = 32.673x^-1.2626
R^2 = 0.9828
y = 17.841e^-0.2137x
R^2 = 0.8881
N(X>W)
£BN
20
18
16
14
12
10
8
6
4
2
0
0
5
10
15
20
2006
Power (2006)
Expon. (2006)


UK Cumulative weekly income 1995
1.5
2.5
3.5
4.5
5.5
6.5
7.5
0
-2
-4
-6
-8
-10
-12
-14
y = -3.2453x + 7.8691
R2 = 0.9997
y = -3.3271x + 7.5344
R2 = 0.9997
y = -1.2626x - 1.335
R2 = 0.9828



0.04
0.03
0.02
0.01
0
P(age)
0
20
40
60
80
100
age
2003
2004
2005
2006

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