A variational approach to wealth distributions

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A Variational Principle for Pareto's power law

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Many-Agent Models with a Mechanical Analogue

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Entropy-based variational principle in econophysics

Variational Lagrange principles, in particular entropy-based approaches, as well as general thermodynamics concepts such thermodynamic cycle, find their natural application in the study of social and economic processes [1].

Example: a one-dimensional system

Vary the entropy $S[f]=\int dq\, f(q)\ln[f(q)]$

with constraint $\int dq\, f(q)=1$ on probability

$\int dq\, f(q)\, X(q)=x_{tot}$ on total energy

Lagrange method: vary

$$S[f]=\int dq\, f(q)[\ln[f(q)]+\mu+\beta x] \quad \rightarrow \quad f(q)=\frac{\exp(-\beta x)}{\langle x \rangle}$$

General idea: study a statistical mixture of systems with different number of degrees of freedom.

[1] E.g. the contributions of J. Mimkes *et al.*, in:
A. Chatterjee, S.Yarlagadda, B. K. Chakrabarti Eds., *Econophysics of Wealth Distributions*, Springer, 2005.

Model with constant saving propensity [1]

Model: N agents: $1, \; 2, \; \ldots, N$
with wealths $x_1, x_2, \ldots, x_N$

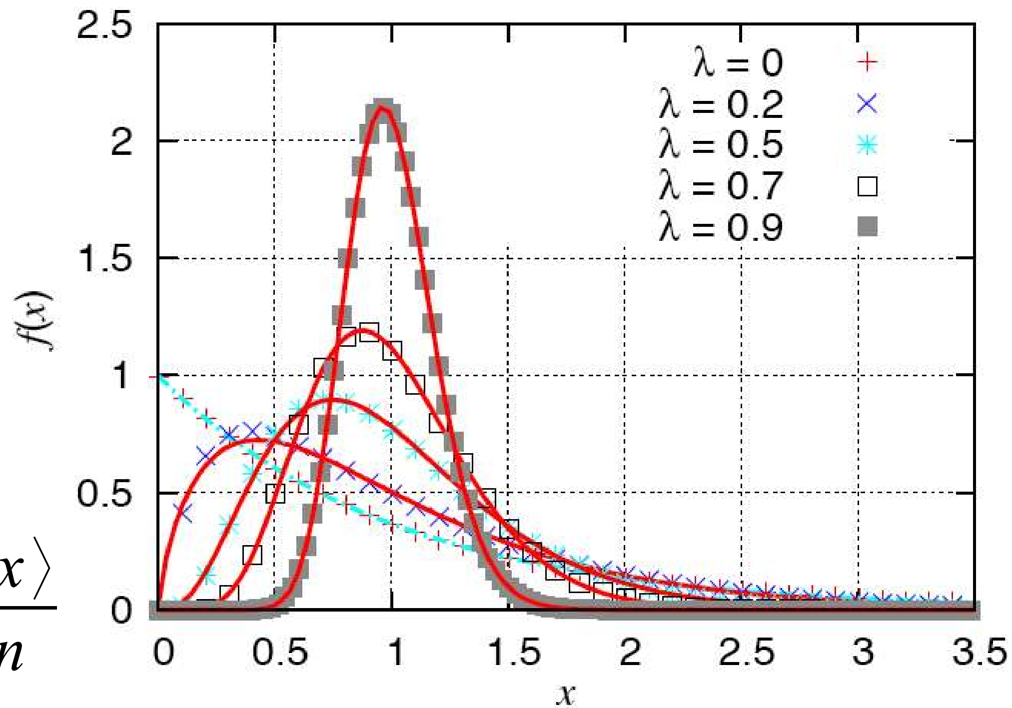
Evolution: $x_i \to \lambda x_i + \epsilon(1-\lambda)(x_j + x_i)$
$x_j \to \lambda x_j + (1-\epsilon)(1-\lambda)(x_j + x_i)$

The equilibrium distribution is well fitted by a **gamma-distribution**

$$f(x) = \beta \gamma_n(\beta x) = \frac{\beta(\beta x)^{n-1}}{\Gamma(n)} \exp(-\beta x)$$

$$n = 1 + \frac{3\lambda}{1-\lambda} \equiv \frac{N}{2}$$

- $N = 2n$ is an effective dimension
- equipartition theorem $\quad \beta^{-1} = \frac{\langle x \rangle}{N/2} \equiv \frac{\langle x \rangle}{n}$



[1] J. Angle, Social Forces 65 (1986) 293.
A. Chakraborti, B. K. Chakrabarti, Eur. Phys. J. B 17 (2000) 167.
M. Patriarca, A. Chakraborti, K. Kaski, Physica A 340 (2004) 334.
M. Patriarca, A. Chakraborti, K. Kaski, Phys. Rev. E 70 (2004) 016104.

Variational approach for constant saving propensity

1) Assuming an **energy** (N-dimensional system): $X(q)=\frac{1}{2}\left[q_1^2+\ldots+q_N^2\right]$

2) Construct the **entropy** (with constraints)

$$S[f]=\int dq_1 dq_2 \ldots f(q_1,q_2,\ldots)\left\{\ln[f(q_1,q_2,\ldots)]+\mu+\beta X(q_1,q_2,\ldots)\right\}$$

3) Integrate the $(N-1)$ angular variables, q = distance in N-dimensional space

$$S[f_1]=\int dq\, f_1(q)\left\{\ln\left[\frac{f_1(q)}{\sigma_N q^{N-1}}\right]+\mu+\beta X(q)\right\}$$

4) Finally obtain the **reduced entropy** as an integral over energy variable

$$S[f]=\int dx\, f(x)\left\{\ln\left[\frac{f(x)}{\sigma_N x^{N/2-1}}\right]+\mu+\beta x\right\}\rightarrow f(x)=\frac{\beta}{\Gamma(n)}(\beta x)^{n-1}\mathrm{e}^{(-\beta x)}$$

Here:

$x=X(q)=q^2/2$ Energy variable

$\sigma_N=2\pi^{N/2}/\Gamma(N/2)$ **surface** of an N-dimensional sphere with radius $q=1$

$f_1(q)=f_N(q)/\sigma_N q^{N-1}$ **One-dimensional density** for the radius coordinate q

Generalized model with different saving propensities

Model [1]

N agents: 1, 2, ..., N
with wealths $x_1, x_2, \ldots, x_N$

Evolution law

$$x_i \to \lambda_i x_i + \epsilon[(1-\lambda_i)x_i + (1-\lambda_j)x_j]$$
$$x_j \to \lambda_j x_j + (1-\epsilon)[(1-\lambda_i)x_i + (1-\lambda_j)x_j]$$

This model is known from numerical simulations and analytical calculations to produce a power law tail in the density,
$f(x) \sim 1/x^2$

[1] J. Angle, J. Math. Soc. 26 (2002) 217.
A. Chatterjee, B. K. Chakrabarti, S. S. Manna, Physica Scripta T106 (2003) 36.
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P. Repetowicz, S. Hutzler, P. Richmond, Physica A 356 (2005) 641.
M. Patriarca, A. Chakraborti, K. Kaski, G. Germano in A. Chatterjee, S.Yarlagadda, B. K. Chakrabarti (Eds.), *Econophysics of Wealth Distributions*, Springer, 2005.

Global Entropy

$$S[f]=\int dn\, P(n)\int dx\, f_n(x)\left\{\ln\left[\frac{f_n(x)}{x^{n-1}}\right]+\mu_n+\beta x\right\}$$

Different constraints (μ_n) for probability conservation $\int_0^\infty dx\, f_n(x)=1$

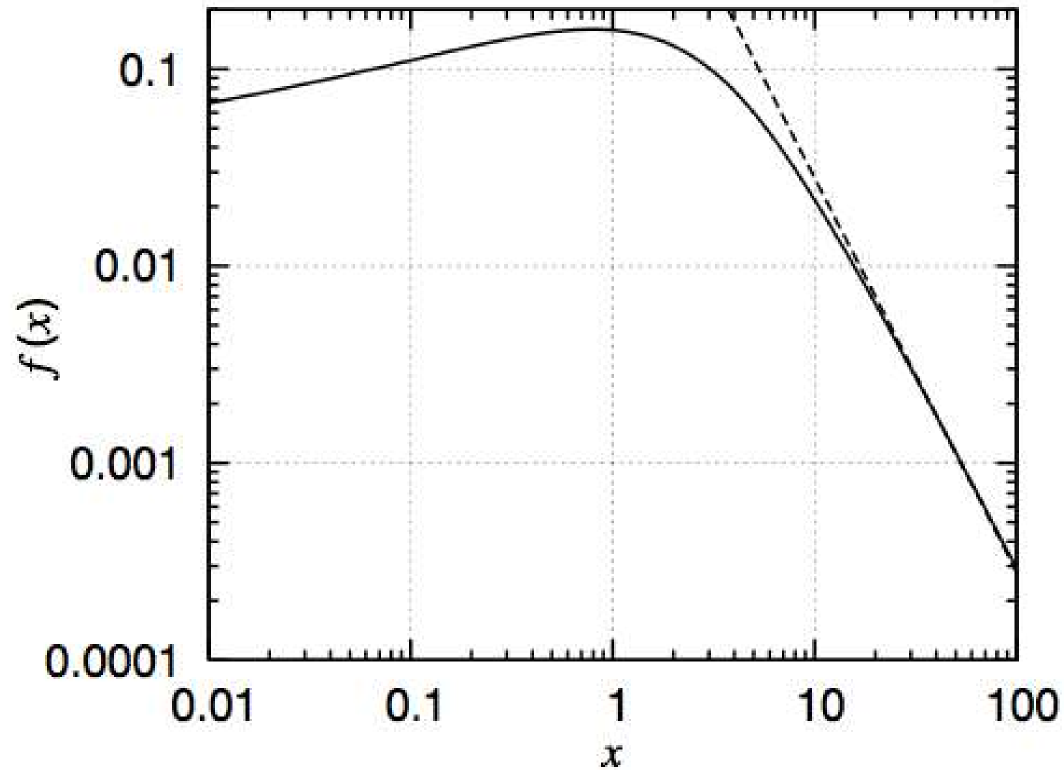
A single constraint on energy conservation $\int dn\, P(n)\int_0^\infty dx\, f_n(x)=1$

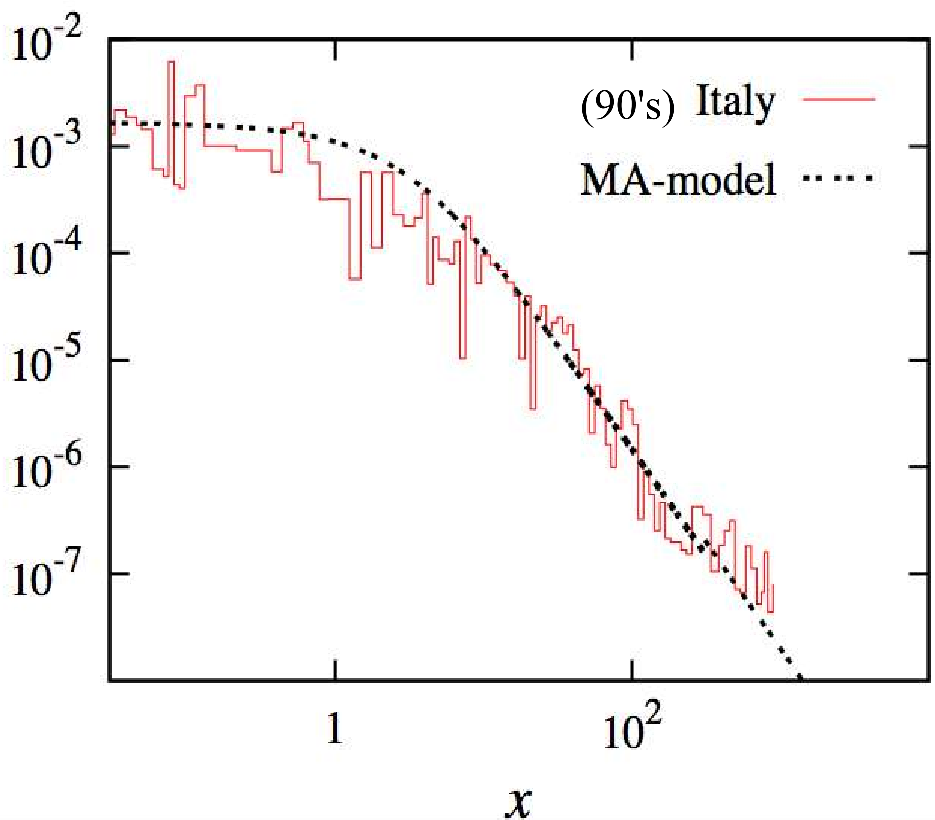
Global density

$$f(x)=\int dn\, P(n)\,\beta\,\gamma_n(\beta x)=\int dn\, P(n)\frac{\beta(\beta x)^{n-1}}{\Gamma(n)}\exp(-\beta x)$$

[1] See e.g. M. Patriarca, A. Chakraborti, K. Kaski, G. Germano in A. Chatterjee, S.Yarlagadda, B. K. Chakrabarti (Eds.), *Econophysics of Wealth Distributions*, Springer, 2005.

General shape of the global distribution $f(x)$





Data from K. Okuyama, M. Takayasu, and H. Takayasu, Physica A 269 (1999) 125.

Final remarks and future directions

• The model of distributed saving propensity only produces power laws in the cumulative distribution going as $1/x$. How to generalize the model and yet keep it simple?

• The saving propensity λ does not reflect the (microscopic) strategy of an investor, rather there is a work of economic interpretation of the model to be done to understand the meaning of λ.

• The small x part of the distribution also is not well reproduced.

• However comparison with distributions with power law tails $f(x) \sim 1/x^2$ is already possible: other examples of wealth distributions?

• The model applies also to other social and mechanical systems.