Modelling long-range memory trading activity by stochastic differential equations

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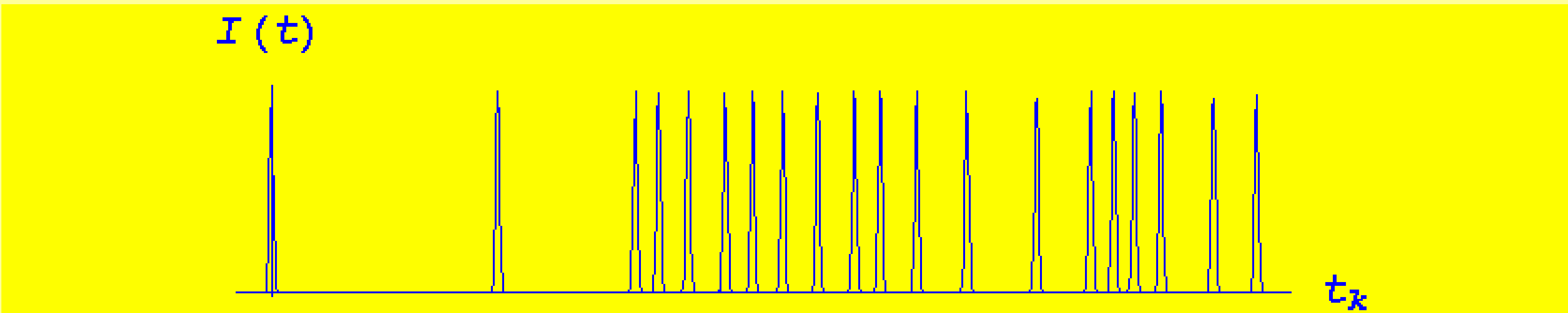
Risk Physics

Outline

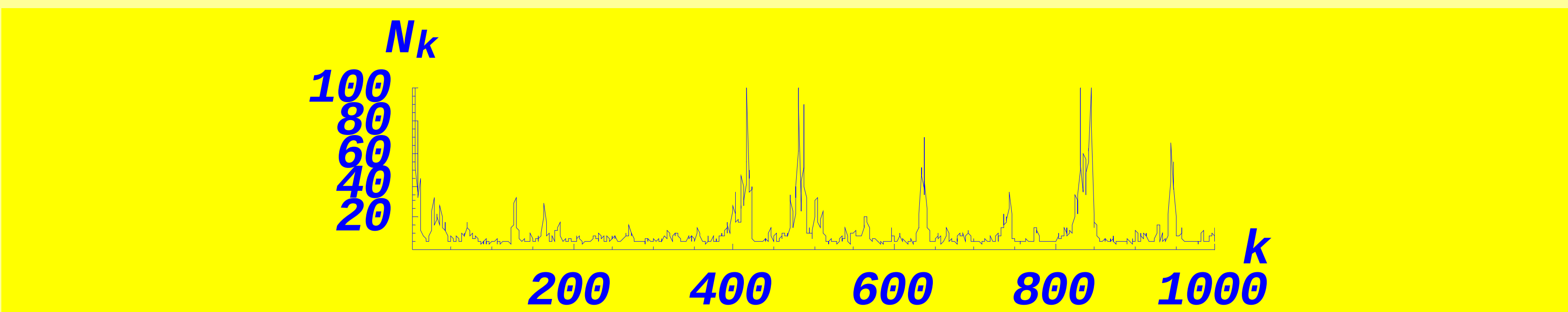
- Flow of trades as a point process with long-range memory
- Background model - Modulated Poisson process
- Stochastic models of trading activity, return and volatility

Signal as a stochastic sequence of pulses

$$I(t) = \sum_k A_k(t - t_k), \;\; A_k(t) = T_k^{\delta} A\left(\frac{t}{T_k}\right)$$



$$\{t_1, t_2, t_3, ..t_k, ..t_n\} \quad or \quad \tau_k = t_k - t_{k-1}$$



Stochastic models of interevent time

1. Poisson processes

$$P(\tau) = \frac{1}{\langle\tau\rangle}\exp(-\frac{\tau}{\langle\tau\rangle}); \quad \beta = 0$$

2. Fractal renewal processes

$$P(\tau) = \frac{\beta}{\tau_{\min}^{-\beta} - \tau_{\max}^{-\beta}}\tau^{-(\beta+1)}$$

3. Autoregressive conditional duration (ACD) processes

$$P(\tau_{k+1}) = \frac{1}{\langle\tau\rangle}\exp(-\tau_{k+1}/\langle\tau\rangle), \qquad \langle\tau\rangle = \tau_0 + \sum_{j=0}^{K} a_j \tau_{k-j}$$

4. Recurrent stochastic point processes

$$\tau_{k+1} = \tau_k + \gamma\tau_k^{2\mu-1} + \sigma\tau_k^{\mu}\varepsilon_k, \quad \beta = 1 + \alpha/(3-2\mu)$$

$$\frac{d\tau_k}{dk} = a(\tau_k) + b(\tau_k)\xi(k)$$

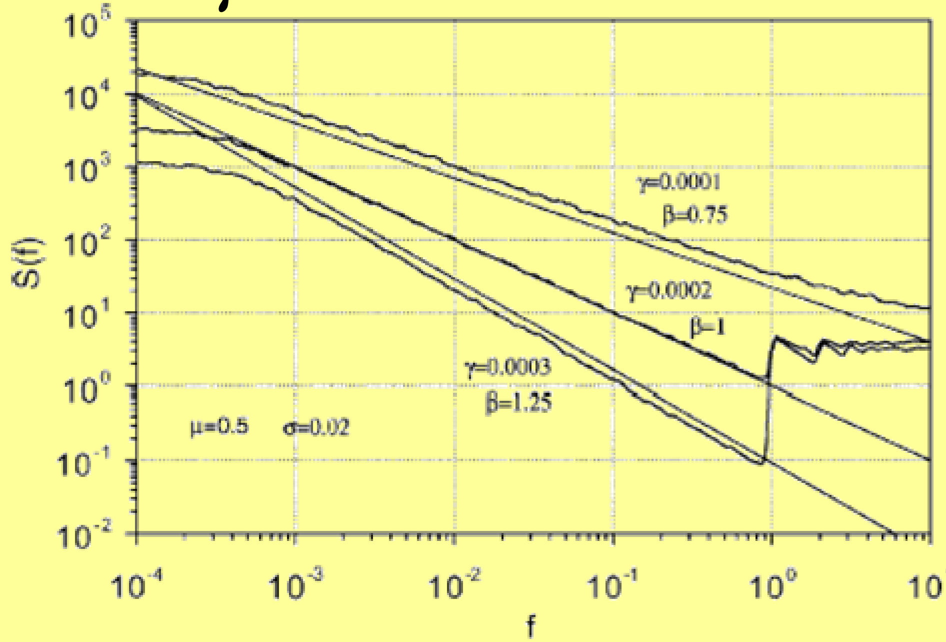
Multiplicative point process

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \tau_k^{\mu} \sigma \varepsilon_k$$

$$P(\tau_k) \sim \tau_k^{\alpha}, \ \ \alpha = 2\gamma / \sigma^2 - 2\mu$$

$$\beta = 1 + \frac{\alpha}{3 - 2\mu}$$

$$P(N) = \begin{cases} \dfrac{1}{N^{3+\alpha}}, N << \gamma^{-1} \\ \dfrac{1}{N^{5+2\alpha}}, N >> \gamma^{-1} \end{cases}$$



V. Gontis, B. Kaulakys, PHYSICA A, (2004)

Model definitions

$$d\tau = [\gamma - \tfrac{m}{2}\sigma^2 (\tfrac{\tau}{\tau_0})^m]\tau^{2\mu-2}dt + \sigma\tau^{\mu-1/2}dW$$

$$P(\tau) \sim \tau^{\alpha} \exp[-(\tfrac{\tau}{\tau_0})^m] \qquad \alpha = 1 + 2\gamma/\sigma^2 - 2\mu$$

$$\varphi(\tau_p \middle| \tau) = \tfrac{1}{\tau}\exp[-\tfrac{\tau_p}{\tau}]$$

$$\varphi(\tau_p) = c\int_0^{\infty} \exp[-\tfrac{\tau_p}{\tau}]\tfrac{1}{\tau^{1-\alpha}}\exp[-(\tfrac{\tau}{\tau_0})^m]d\tau$$

$$\varphi(\tau_p) = \frac{2}{\Gamma(1+\alpha)\tau_0^{\alpha+\frac{1}{2}}\tau_p^{\alpha+\frac{1}{2}}} K_{-\alpha}\left(2\sqrt{\frac{\tau_p}{\tau_0}}\right)$$

Flow of events or trades

$$dn = \frac{\sigma^2}{\tau_d}[(1-\gamma_\sigma) + \frac{m}{2}(\frac{n_0}{n})^m]n^{2\eta-1}dt + \frac{\sigma}{\tau_d^{1/2}}n^\eta dW$$

$$n = \frac{\tau_d}{\tau} \qquad where\ \eta = \frac{5}{2} - \mu \ \ and \ \ n_0 = \frac{\tau_d}{\tau_0}$$

$$P(n) \sim \frac{1}{n^\lambda}\exp\left\{-\left(\frac{n_0}{n}\right)^m\right\}, \quad \lambda = 2(\eta - 1 + \gamma_\sigma)$$

$$S(f) \sim \frac{1}{f^\beta}, \quad \beta = 2 - \frac{3-2\gamma_\sigma}{2\eta - 2}$$

$$Empirical\ values\ of\ \ \beta = 0.7 \ \ and \ \ \lambda = 4.4$$

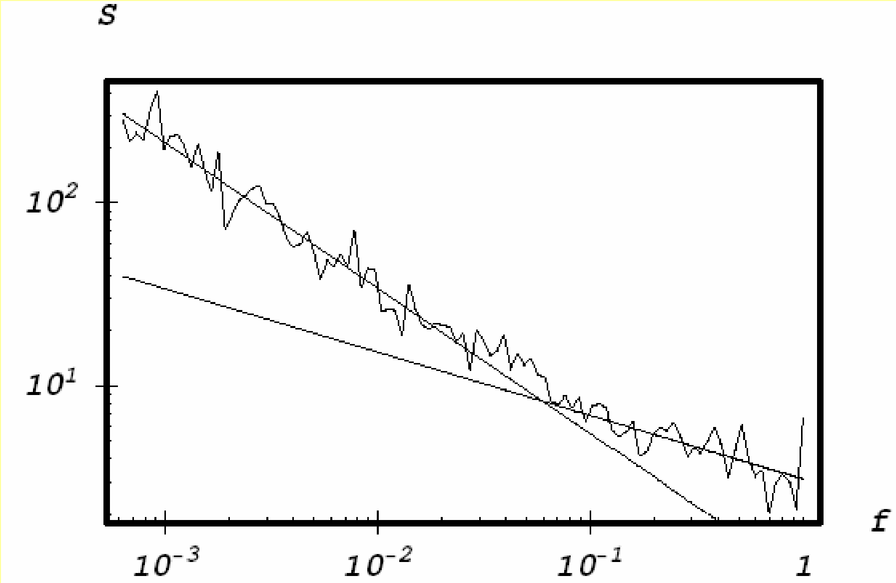
Stochastic model of trading activity

$$N = \frac{1}{\tau_d} \int_t^{t+\tau_d} n(s)ds \quad P(N) \sim \begin{cases} \frac{1}{N^{3+2\gamma_\sigma - 2\mu}}, & N << \gamma^{-1} \\ \frac{1}{N^{5+2(\gamma_\sigma - 2\mu)}}, & N >> \gamma^{-1} \end{cases}$$

$$dn = \frac{\sigma^2}{\tau_d}\left[(1-\gamma_\sigma) + \frac{m}{2}\left(\frac{n_0}{n}\right)^m\right]\frac{n^4}{(n\varsigma + \tau_d)^2}dt + \frac{\sigma}{\tau_d^{1/2}}\frac{n^{5/2}}{(n\varsigma + \tau_d)}dW$$

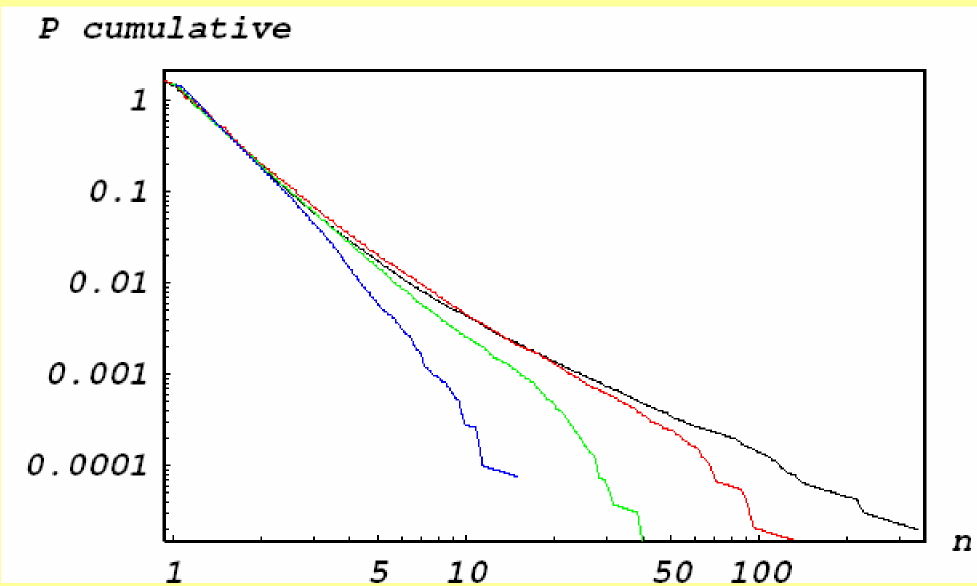
$$\tau_{k+1} = \tau_k + \left[\gamma - \frac{m}{2}\sigma^2\left(\frac{\tau}{\tau_0}\right)^m\right]\frac{\tau_k}{(\varsigma + \tau_k)^2} + \sigma\frac{\tau_k}{\varsigma + \tau_k}\varepsilon_k$$

$$\gamma = 0.0004; \quad \sigma = 0.025; \quad \varsigma = 0.07; \quad \tau_0 = 1; \quad m = 6;$$



$$S \sim \frac{1}{f^{\beta_i}}$$

$$\beta_1 = 0.35 \quad and \quad \beta_2 = 0.8$$



Distribution of **n**

Distribution of **N**

$\tau_d = 10$

$\tau_d = 50$

$\tau_d = 250$

Power spectral density of trading activity

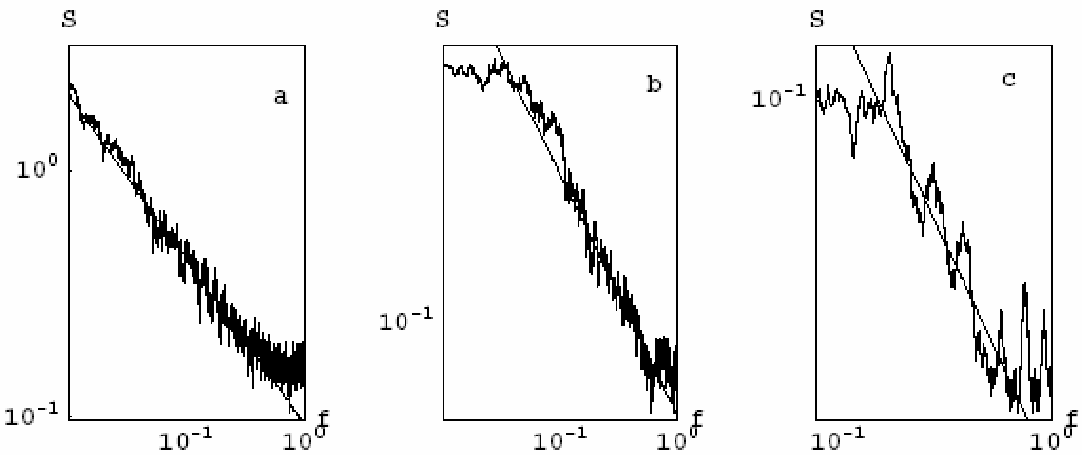
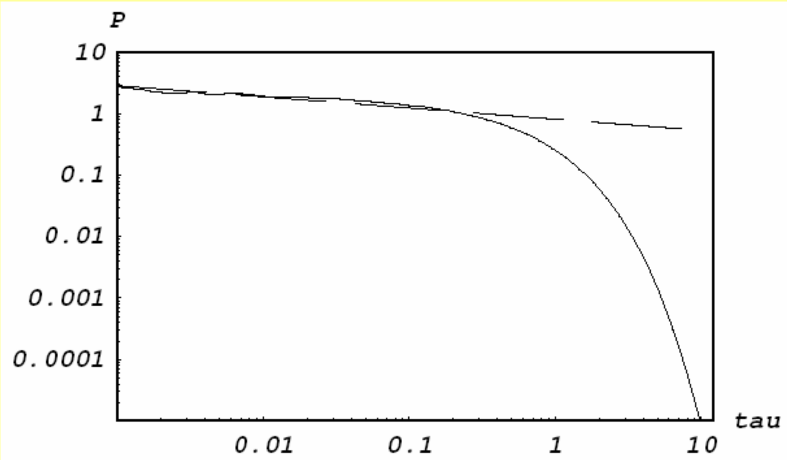


Figure 3. Power spectral density calculated by the Fast Fourier Transform of N series generated with (12) for the same parameters as in figures 1 and 2: a) $\tau_{\rm d} = 10$; b) $\tau_{\rm d} = 50$; c) $\tau_{\rm d} = 250$. Straight lines approximate power spectrum $S \sim 1/f^{\beta}$, where $\beta = 0.7$.

Waiting time distribution

$$d\tau = \left[\gamma - \frac{m}{2}\sigma^2\left(\frac{\tau}{\tau_0}\right)^m\right]\frac{1}{(\varsigma+\tau)^2}dt + \sigma\frac{\sqrt{\tau}}{\varsigma+\tau}dW$$



$$\varphi(\tau_p \mid \tau) = \frac{1}{\tau}\exp\left[-\frac{\tau_p}{\tau}\right]$$

$$P(\tau_p) \sim \tau_p^{-0.15}$$

Ivanov P.Ch, Yuen A., Podobnik B., Lee Y., PHYSICAS REVIEW E **69**. 056107 (2004)

Modeling volatility and return

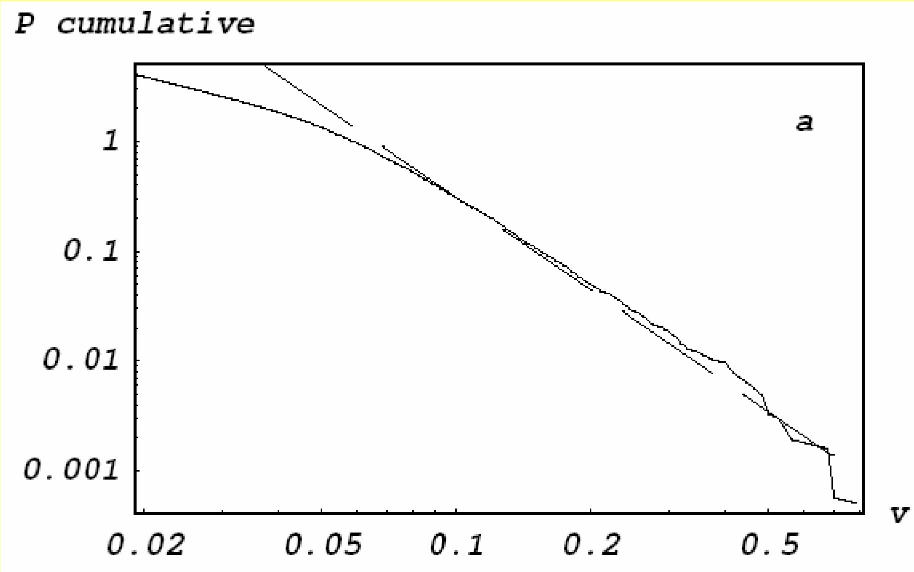
Plerou V., Gopikrishnan P., Gabaix X., Amaral L., Stanley H.E., QUANTATIVE FINANCE (2001)

$$x(t,\tau_d) = \sum_{i=1}^{N(t,\tau_d)} \delta p_i \qquad x(t,\tau_d) = w(t,\tau_d)\sqrt{N(t,\tau_d)}\varepsilon_t$$

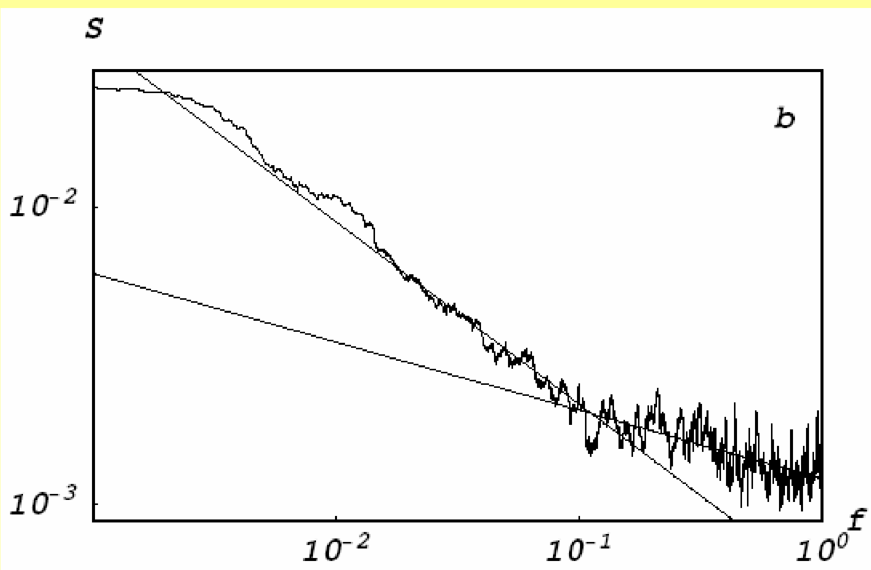
$$w(t,\tau_d) = kn(t) \qquad x(t,\tau_d) = kn(t)\sqrt{N(t,\tau_d)}\varepsilon_t$$

$$v(t,\tau_d) = \left|x(t,\tau_d)\right| \qquad \bar{v}_k = \frac{1}{m}\sum_{i=k}^{i=k+m} v(t_i,\tau_d)$$

Farmer J. Doyne et al, PHYSICAL REVIEW LETTERS **90**, No10, (2003)



$$P(\bar{v}_k) \sim \frac{1}{\bar{v}_k^{\,2.8}}$$



$$S \sim \frac{1}{f^{\beta_i}}$$

$$\beta_1 = 0.23 \quad and \quad \beta_2 = 0.6$$

Conclusions:

- We proposed a stochastic differential equation as a dynamical model of the observed memory in financial time series
- The continuous stochastic process reproduces statistical properties of trading activity and serves as a background model for the modeling waiting time, return and volatility
- Empirically observed statistical properties: exponents of power-law probability distributions and power spectral density of long-range memory financial variables are reproduced with the same value of the main model parameter:

$$\gamma_\sigma = \frac{\gamma}{\sigma^2}$$