Ising model on two connected Barabasi-Albert networks

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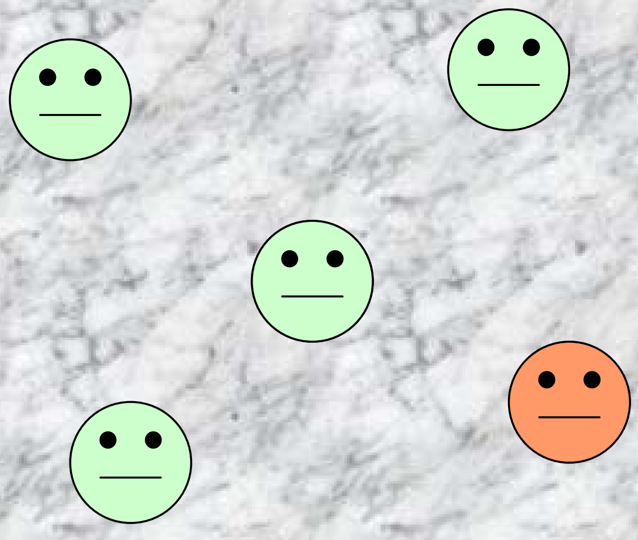
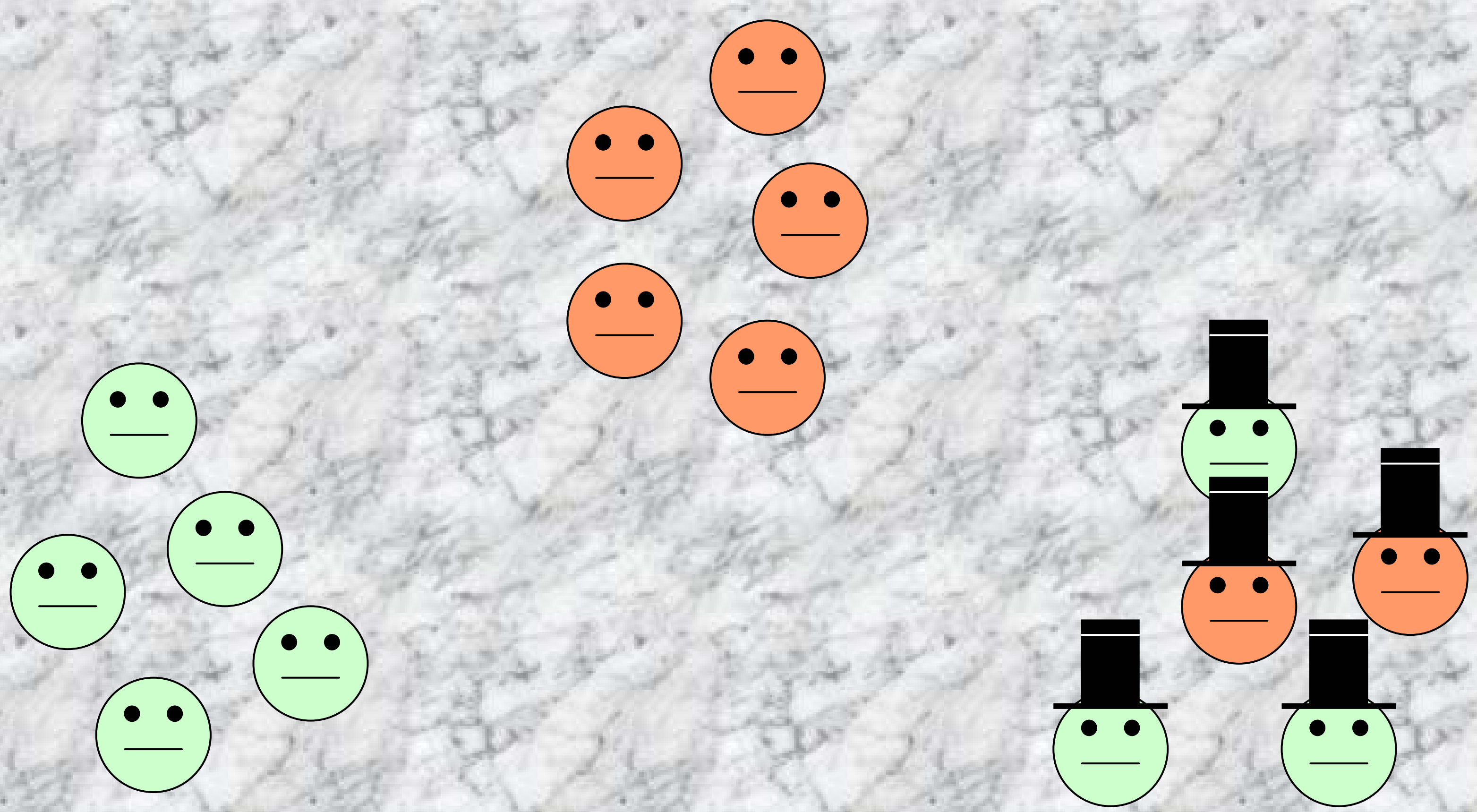
Introduction

- Social interactions – often majority-driven



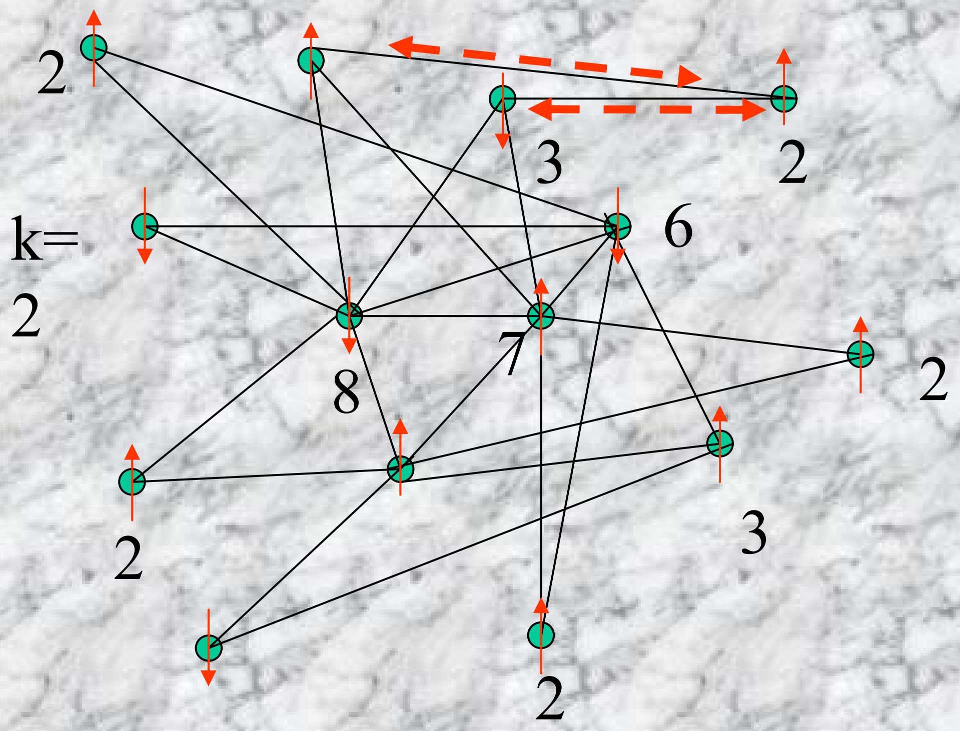
Introduction

- Social interaction networks are often modular



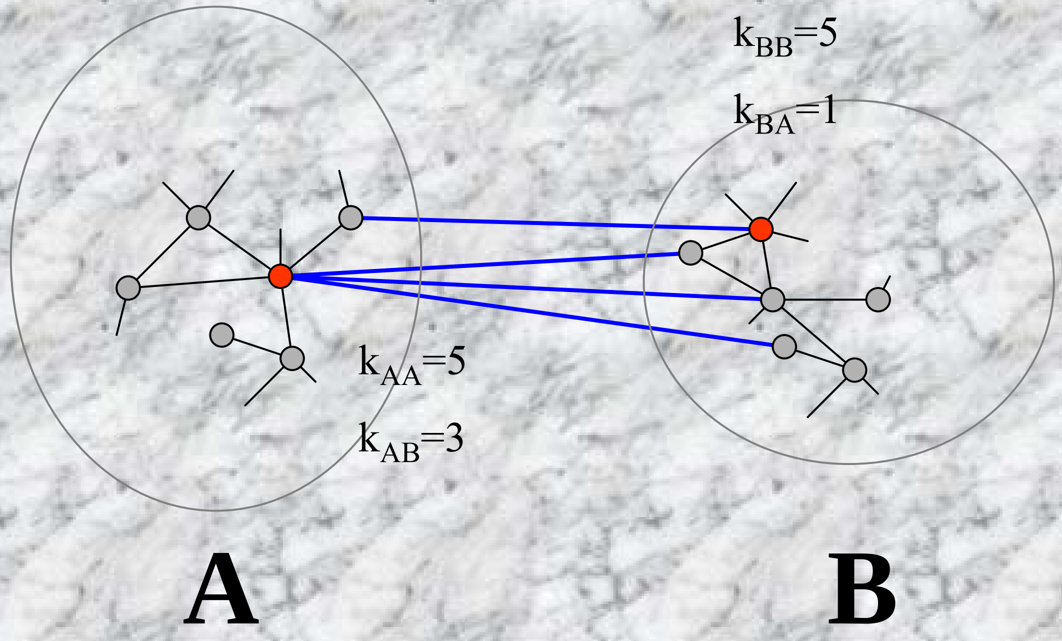
Introduction

- We represent simplest social interactions with Ising model on network



Introduction

- And then study coupling of two networks (one larger modular network)



Regular networks

Hamiltonian for the Ising model, assuming the interaction constant is same for all pairs of spins.

$$H = -J \sum_{i,j} s_i s_j$$

Equation for mean spin (magnetization) in the mean-field approximation:

$$\langle s_i \rangle = \tanh\left(\beta \sum_j J_{ij} s_j \right) = \tanh\left(\beta J k \langle s_i \rangle \right)$$

k - coordination number (number of neighbors)

Scale-free networks

$$\langle s_i \rangle = \tanh\left(\beta \sum_j J_{ij} s_j \right) = \tanh\left(\beta J \sum_j \varepsilon_{ij} \langle s_j \rangle \right)$$

Instead of coincidence matrix ε_{ij} we take the statistical probability of the spins being neighbors:

$$\langle \varepsilon_{ij} \rangle = \frac{k_i k_j}{E} = \frac{k_i k_j}{\langle k \rangle N}$$

We obtain following

$$\langle s_i \rangle = \tanh\left(\frac{\beta J k_i}{\langle k \rangle N} \sum_j k_j \langle s_j \rangle \right)$$

Scale-free networks

$$\langle s_i \rangle = \tanh\left(\frac{\beta J k_i}{\langle k \rangle N} \sum_j k_j \langle s_j \rangle \right)$$

Now we define average weighted spin S as:

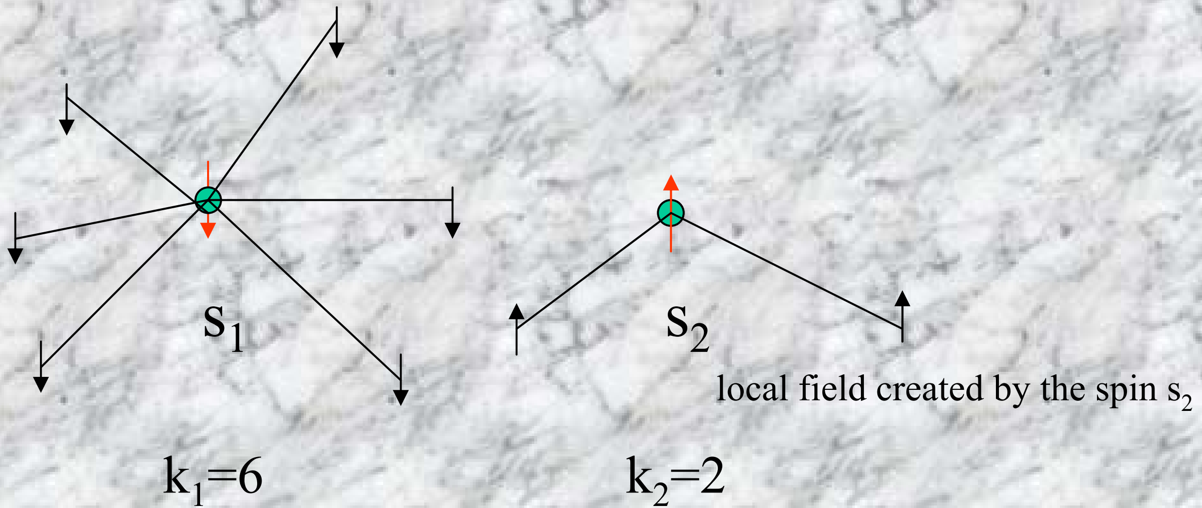
$$S = \frac{1}{\langle k \rangle N} \sum_i k_i \langle s_i \rangle$$

so we can write above as

$$S = \frac{1}{\langle k \rangle N} \sum_i k_i \tanh\left(\beta J k_i S \right)$$

Scale-free networks

What does the weighted spin mean, and why weighted ?



$s_1+s_2=0$ no order ?

$s_1k_1+s_2k_2<0$ order !

Scale-free networks

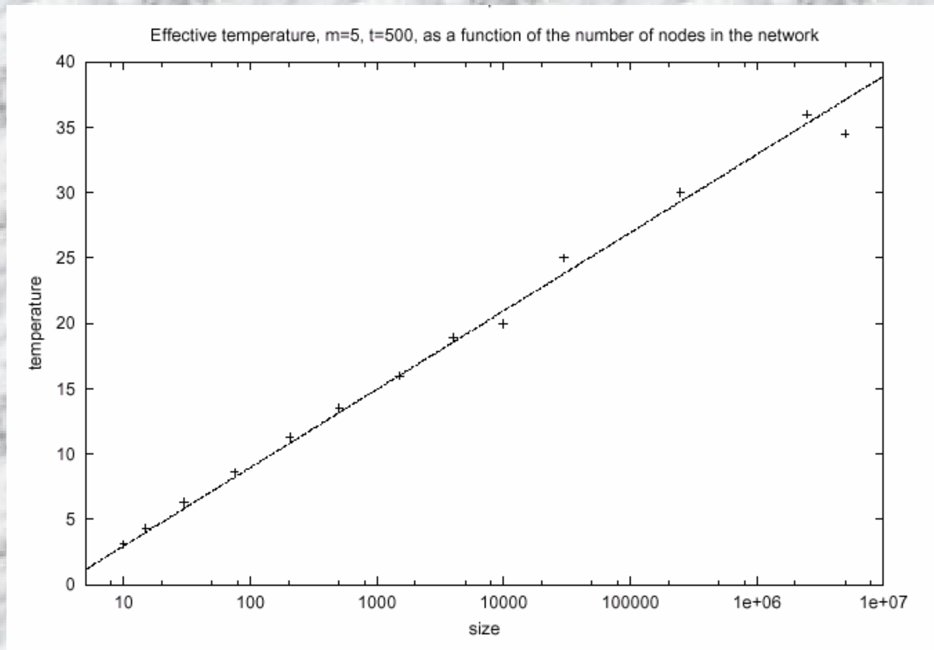
$$S = \frac{1}{\langle k \rangle N} \sum_i k_i \tanh\left(\beta J k_i S \right)$$

Linear approximation

$$S = \frac{1}{\langle k \rangle N} \sum_i k_i^{\,2} \beta J S = \beta J \, \frac{\sum_i k_i^{\,2}}{\langle k \rangle N} S = \beta J \, \frac{\langle k^2 \rangle}{\langle k \rangle} S$$

B-A network

$$\beta J \, \frac{\langle k^2 \rangle}{\langle k \rangle} S = \beta J \, \tfrac{m}{2} \ln N S \longrightarrow T_c = J \, \tfrac{m}{2} \ln N$$



Effective Tc versus m+ N for m = 5 and various N, averaged over up to 1000 samples.

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Ferromagnetic phase transition in Barabási–Albert networks

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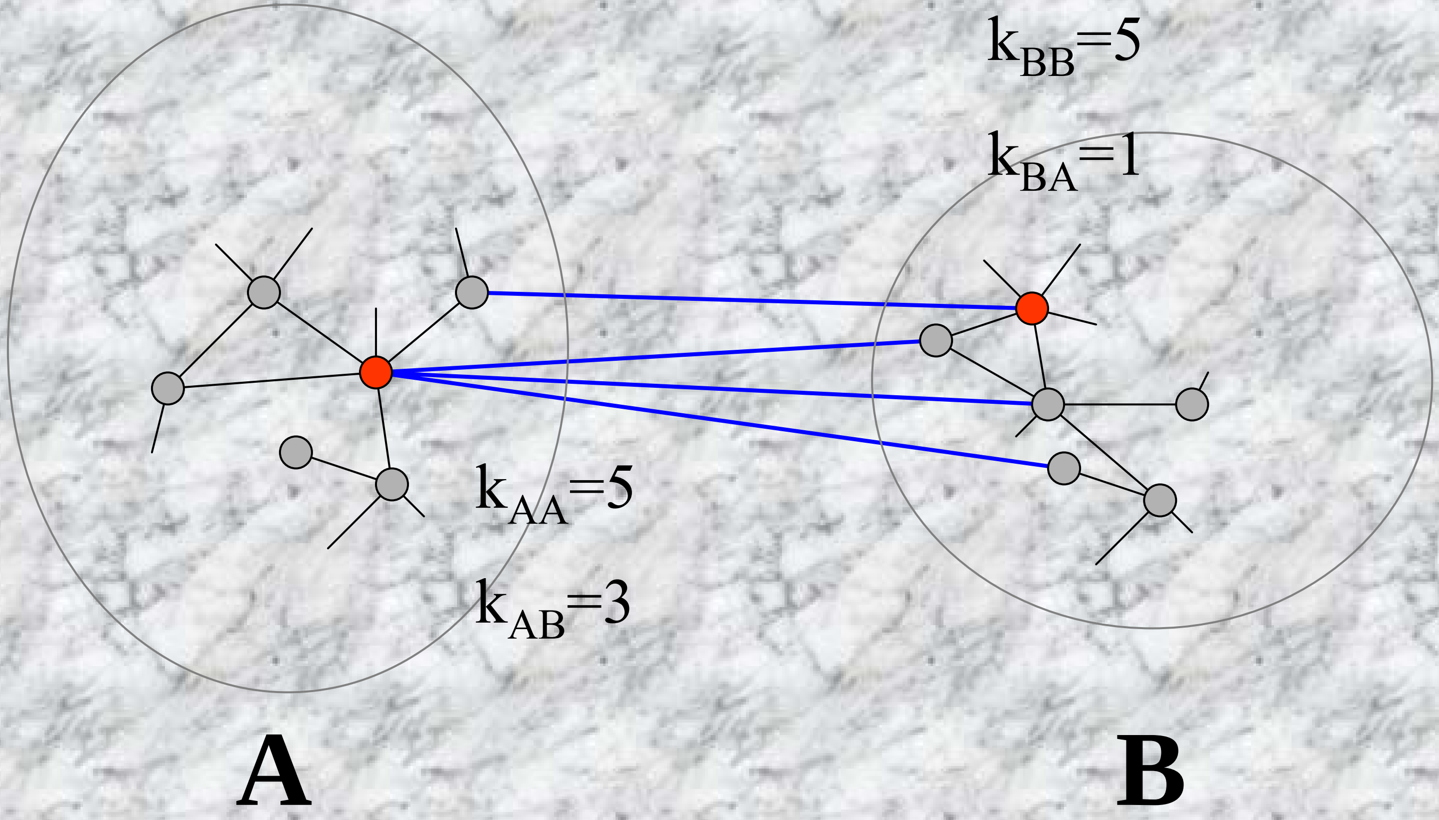
Coupled networks



Coupled networks

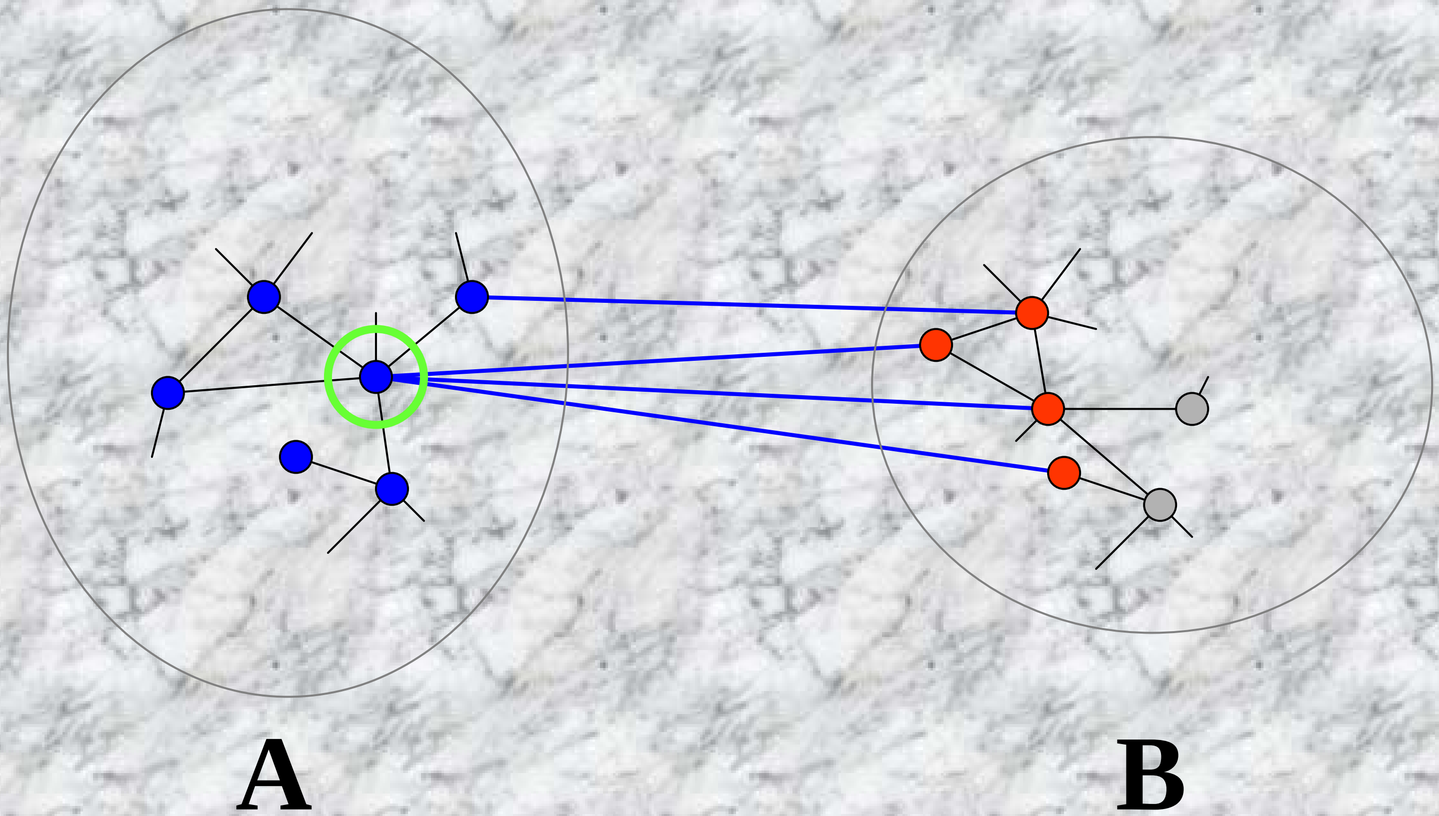
Using linear approximation we investigate existence of nonzero solutions, that correspond to an ordered phase.

$$\langle s_{Ai} \rangle = \beta J_{AA} k_{AAi} \sum_j \frac{k_{Aj} \langle s_{Aj} \rangle}{E_{AA}} + \beta J_{BA} k_{ABi} \sum_j \frac{k_{BAj} \langle s_{Bj} \rangle}{E_{BA}}$$

$$\langle s_{Bi} \rangle = \beta J_{BB} k_{BBi} \sum_j \frac{k_{Bj} \langle s_{Bj} \rangle}{E_{BB}} + \beta J_{AB} k_{BAi} \sum_j \frac{k_{ABj} \langle s_{Aj} \rangle}{E_{AB}}$$

Similar to single scale-free network we introduce weighted spins $S_{.A.}$, S_B.Unfortunately we also need S_{AB} and $S_{BA.}$

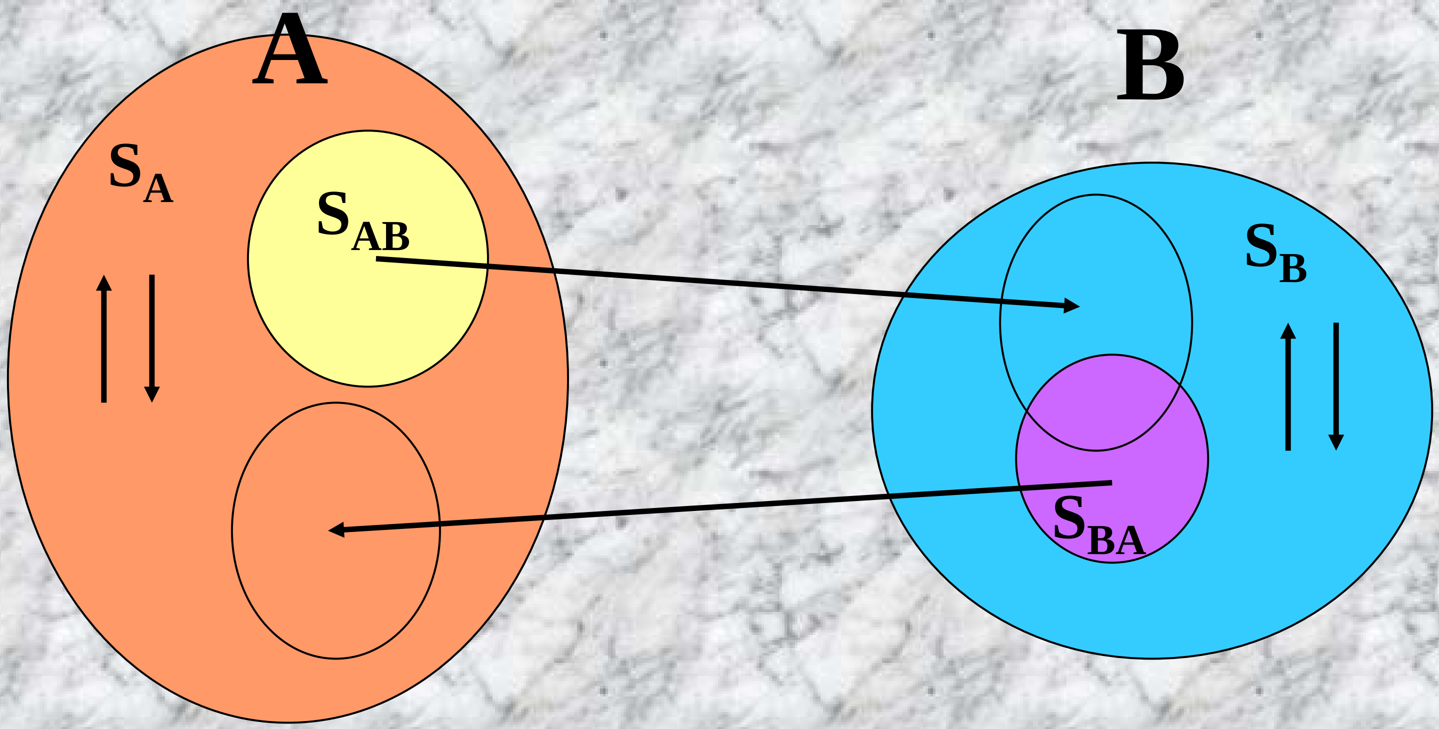
Coupled networks



S_A - spin of network A weighted by k_{AA} (●)

S_{BA} - spin of network B weighted by k_{BA} (●)

Coupled networks



The state of the system can be written as vector: $\begin{pmatrix} S_A \\ S_{AB} \\ S_{BA} \\ S_B \end{pmatrix}$

Coupled networks

If we assume that the number of inter-network connections is proportional to the intra-network degree:

$$k_{AB} = p_A k_{AA}, \qquad k_{BA} = p_B k_{BB}$$

then we don not have to consider S_{AB} and S_{BA} anymore since they are proportional to S_A and S_B.

$$\begin{bmatrix} \Lambda_{AA} & \Lambda_{BA} \\ \Lambda_{AB} & \Lambda_{BB} \end{bmatrix} \begin{pmatrix} S_A \\ S_B \end{pmatrix} = \lambda \begin{pmatrix} S_A \\ S_B \end{pmatrix}$$

Coupled networks

We have following eigenvalues λ of the matrix Λ:

$$\lambda_{\pm} = \frac{\Lambda_{AA} + \Lambda_{BB} \pm \sqrt{(\Lambda_{AA} - \Lambda_{BB})^2 + 4\Lambda_{BA}\Lambda_{AB}}}{2}$$

λ_- : eigenvector $\begin{pmatrix} 1 \\ -c_1 \end{pmatrix}$

The netwoks are ordered antiparalelly. T_C is lower than for separate networks.

Increasing inter-network connection strengths causes T_C to decrease, down to 0, when the inter-network connections are as dense as intra-network.

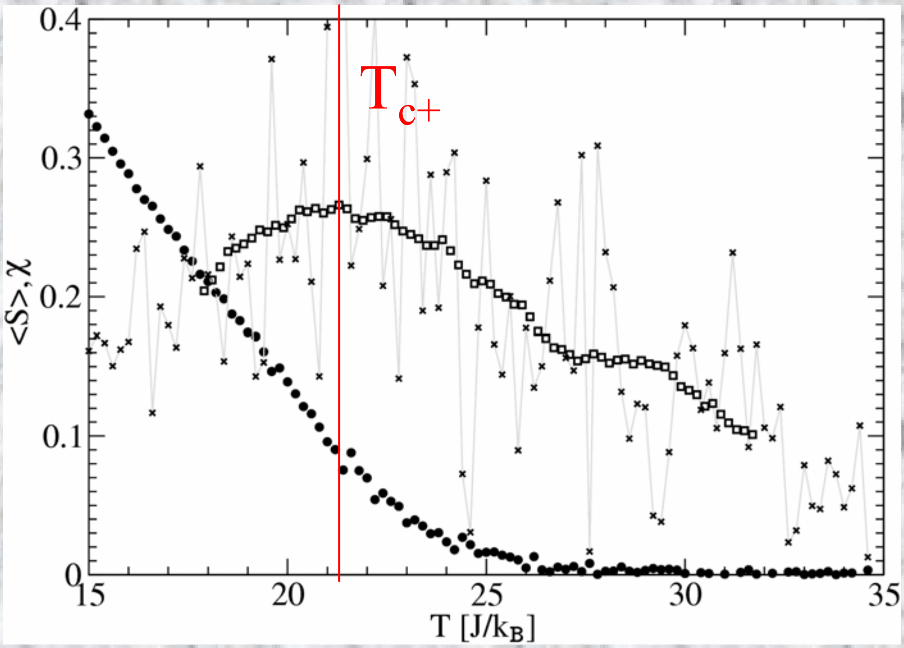
λ_+ : eigenvector $\begin{pmatrix} 1 \\ c_2 \end{pmatrix}$

The networks are ordered paralelly. T_C is higher than for separate networks.

Networks stabilize each other, similar to way the network stabilizes itself.

Numeric results

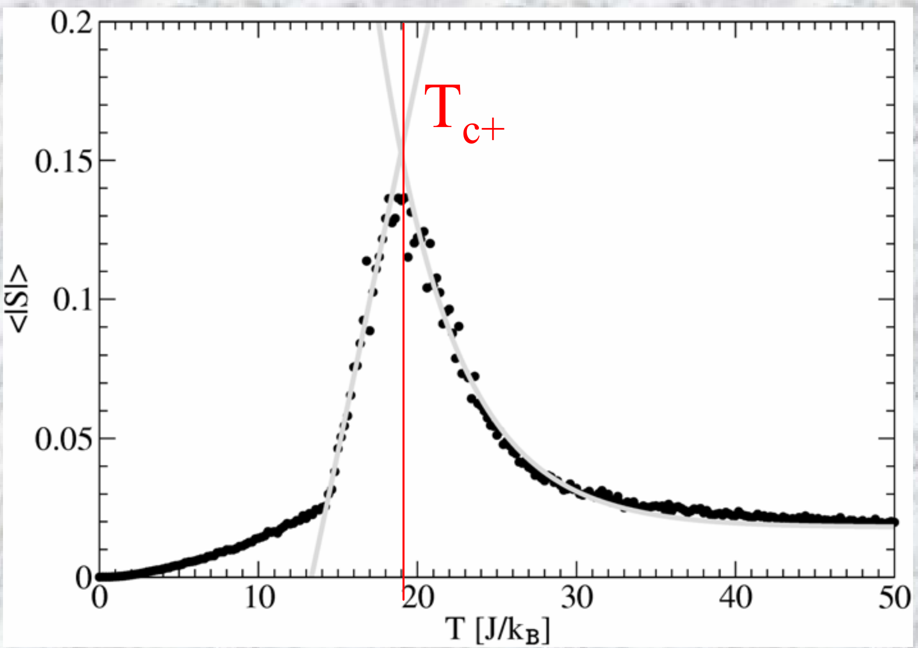
We have performed numeric simulations to find the critical temperatures T_{c+} and T_{c-}.



To find the critical temperature T_{c+} we calculate numerically the susceptibility $\chi=S_h-S_0$. Due to very highly fluctuating nature we make 30-point running average and fit parabolic curve.

Numeric results

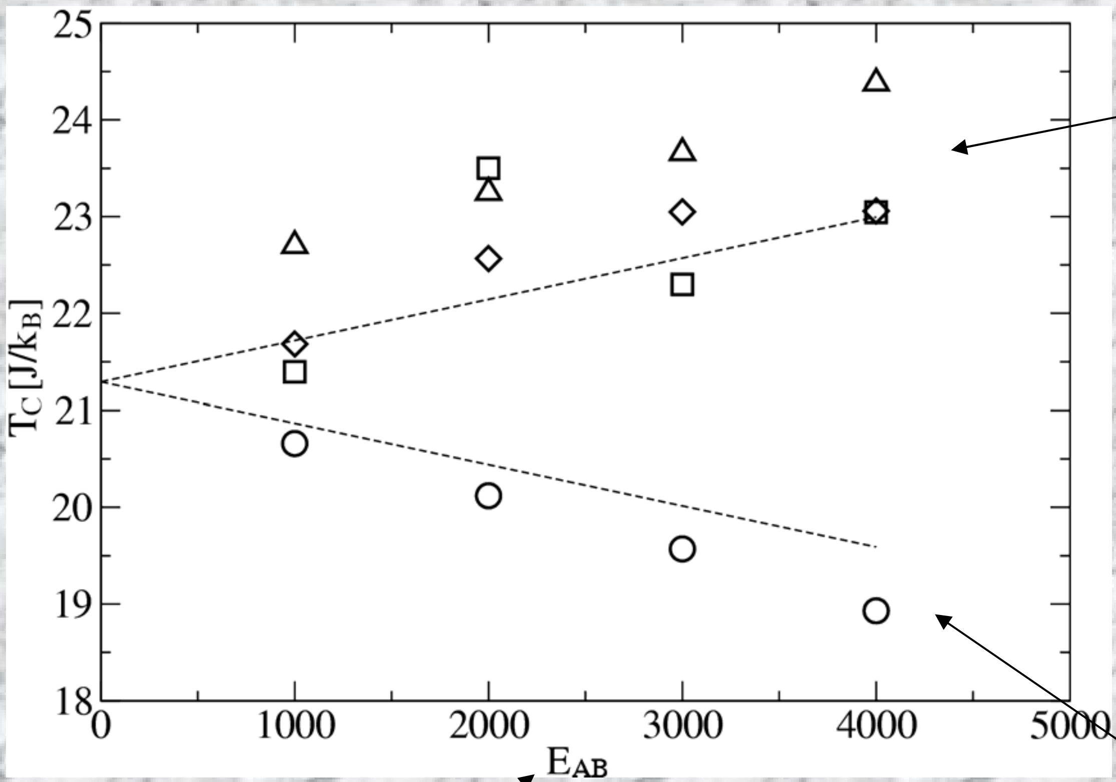
We have performed numeric simulations to find the critical temperatures T_{c+} and T_{c-}.



Plot the total spin starting from antiparallel ordered state as a function of temperature.

Numerical results

Numerical simulations for two same B-A (N=5000, <k>=10, D=0) and various number of inter-network connections:



T_{c+} ,parallel ordering, T_{c+}=A+C

Symbols - numeric results

Lines - analytic formulas

T_{c-} ,antiparallel ordering, T_{c-}=A-C

Inter-network link number

Questions

•What other dynamics can be investigated on connected networks and can they be described in the same way ?

•What other network topologies are worth investigating ?

Ref.: K. Suchecki, J.A. Holyst
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